Particle transport in porous media: The role of inertial effects and path tortuosity in the velocity of the particles

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Fluid in porous media flows through tortuous paths. If the size of the solid particles suspended in the fluid and the fluid velocities are large enough, Brownian motion effects may not be dominant. In this parameter regime, the average velocity of the particles is different than that of the fluid. We obtain the relation between the average velocity of the particles and the average velocity of the fluid in the context of a simple mathematical model and we apply our results to flows near wells. © *2009 American Institute of Physics*. [doi:10.1063/1.3263718]

A porous medium is a material that contains spaces filled with fluid embedded in a solid matrix. These fluid filled spaces are called pores or voids. Soils are examples of porous media.

Suspensions are fluids with suspended small solid particles that we call fine particles. When the particles are small enough. Brownian motion causes the particles to move with an average velocity equal to the average velocity of the fluid. However, this may no longer be the case when the particles are larger.

Fluid flow through porous media takes place along tortuous rather than straight paths. Thus, if inertial effects outweigh Brownian motion effects, the tortuosity of flow paths will cause particles to collide with pore walls as they travel with the fluid. After each collision, a particle loses momentum and needs to be accelerated again by drag forces. As a result, the average velocity of particles may be smaller than that of the fluid. In this letter, we develop a mathematical model to study this phenomenon and discuss its implications to flows near wells. Our results are relevant in other fields where the transport of particles in porous media is of importance including transport of contaminants in soils, transport of nutrients in bones, man made filters, and biological filters.

Two messages we hope to convey with this letter: (1) The average velocity of the particles is sometimes different than the average velocity of the fluid and (2) in this parameter regime, a fluid flow that is not homogeneous in space will lead to a localization of particle concentration, i.e., the particle concentration will increase in certain regions in space. This may lead to the clogging of part of the medium.

The dynamics of particles in porous media is very complex and many physical effects are important.[1] Our analysis is based on a simple model to gain understanding on the role of inertia and path tortuosity in the velocity of the particles while neglecting other physical effects that may also be important.

We start with a basic review of fluid mechanics. Assume that an incompressible spherical particle with radius r_p is immersed in an incompressible Newtonian fluid that extends to infinity. Assume also that the particle moves with constant velocity $\mathbf{u}$ and the velocity of the fluid tends to the constant value $\mathbf{v}$ far away from the particle. It is well known (see Ref. 2) that the *drag* force the fluid exerts on the particle is $\mathbf{F}=6\pi r_p \mu(\mathbf{v}-\mathbf{u})$, where μ is the fluid viscosity.

Consider now a straight channel of length ℓ filled with an incompressible fluid. As an approximation, assume that the fluid velocity is constant in space through the channel. Let v^* be the fluid speed. At time $t=0$, we place a particle of radius r_p and density ρ_p at the upstream end of the channel. Let $x(t)$ be the distance between the particle and upstream end of the channel at time t. We approximate F, the force the particle experiences, by the drag force it would experience if the particle were immersed in a fluid that extends to infinity in all directions, i.e., $\mathbf{F}=6\pi r_p\mu(v^*-x')\mathbf{e}$, where $\mathbf{e}$ is the unit vector parallel to the channel that points downstream. Thus, the particle will move downstream according to Newton's law $(4/3)\pi\rho_p r_p^3 x''=6\pi r_p\mu(v^*-x')$ and $x(0)=x'(0)=0$. This initial value problem can be solved explicitly. In particular, the time T when the particle reaches the downstream end is given implicitly as the solution of

$$\ell = v^\star T - \frac{v^\star}{\kappa}(1-e^{-\kappa T}) \quad \text{where } \kappa = \frac{9}{2}\frac{\mu}{\rho_p r_p^2}. \tag{1}$$

We denote by u^* the average speed of the particle as it travels through the channel. Our first goal is to find u^* as a function of the fluid speed v^* and the parameters of the system. This relation is obtained from Eq. (1) once we note that $u^*=\ell/T$ and replace T by ℓ/u^* in Eq. (1). After simple manipulations we get

$$1 = \frac{v^\star}{\kappa\ell}\left[\frac{\kappa\ell}{u^\star} - (1-e^{-\kappa\ell/u^\star})\right]. \tag{2}$$

Assume now that a suspension flows through the void space of a porous medium. Let $\mathbf{x}_f(t)$ be the path of an element of fluid. This path will not be straight, it will be tortuous. Thus, the distance traveled by the fluid element in a time interval (t_1,t_2), which is $\int_{t_1}^{t_2}\|\mathbf{x}_f'(t)\|dt$, where $\|\cdot\|$ denotes the Euclidean norm, will be larger than the distance from $\mathbf{x}_f(t_1)$ to $\mathbf{x}_f(t_2)$, i.e., $\|\mathbf{x}_f(t_2)-\mathbf{x}_f(t_1)\|^{-1}\int_{t_1}^{t_2}\|\mathbf{x}_f'(t)\|dt \geqslant 1$. We refer to the average value of this ratio over all fluid elements and time intervals (t_1,t_2), as the tortuosity τ. Note that in the literature, tortuosity is often defined as the square of the tortuosity as

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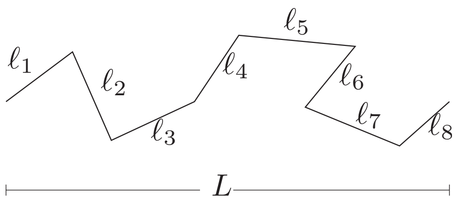

FIG. 1. The small segments form a typical path traveled by an element of fluid in a porous medium.

defined here. This concept is illustrated in Fig. 1. If a typical path traveled by an element of fluid in a porous medium is through the segments with length $\ell_i(1\leq i\leq 8)$, and the distance from the initial to the final positions of the element of fluid is L, then the tortuosity τ is $\tau=(\Sigma_{1\leq i\leq 8}\ell_i)/L$.

We denote by $\mathbf{v}$ the macroscopic fluid velocity, i.e., $\mathbf{v}$ is the average (over the pore space) of the fluid velocity in regions much larger than the pores but much smaller than the material. This velocity $\mathbf{v}$ should not be confused with the Darcian flow velocity $\mathbf{v}_D$. The relationship between these two quantities is $\mathbf{v}=\mathbf{v}_D/\phi$, where ϕ is the porosity, i.e., volume fraction occupied by the pore space.

Let v^* be the average fluid speed. Since the paths traveled by fluid elements are not straight lines, the average fluid speed is larger than the norm of the macroscopic fluid velocity. In fact, we have $v^*=\tau v$ where $v=\|\mathbf{v}\|$. In other words, the ratio between the average of the *microscopic* fluid speed and the norm of the macroscopic fluid velocity is the tortuosity.

Analogously, we denote by $\mathbf{u}$ the macroscopic velocity of the fine particles, i.e., $\mathbf{u}$ is the average of the velocities of the fine particles in regions much larger than the pores but much smaller than the material size, and we denote by u^* the average speed of the fine particles. We assume that we also have $u^*=\tau u$, where $u=\|\mathbf{u}\|$.

Note that these velocities and speeds are, in general, functions of the spatial position $\mathbf{x}$ and time t, i.e., $\mathbf{v}=\mathbf{v}(\mathbf{x},t)$, $\mathbf{u}=\mathbf{u}(\mathbf{x},t)$, $v^*=v^*(\mathbf{x},t)$, and $u^*=u^*(\mathbf{x},t)$.

Our goal is to find $\mathbf{u}$ as a function of $\mathbf{v}$. We assume, naturally, that $\mathbf{u}$ has the same direction as $\mathbf{v}$. We also assume that the typical path a particle follows is as in Fig. 1 with all the segments having the same length ℓ. We also assume that each time a particle reaches the end of one of the segments, it collides with the pore wall and loses all its momentum. Under these modeling conditions, Eq. (2) provides a relationship between the speeds u^* and v^*, where ℓ should be taken as a typical pore size. Thus, given that $v^*=\tau v$ and $u^*=\tau u$, we have

$$1=\frac{v}{\alpha}\left[\frac{\alpha}{u}-(1-e^{-\alpha/u})\right]\quad \text{where } \alpha=\frac{\kappa\ell}{\tau}=\frac{9}{2}\frac{\mu\ell}{\tau\rho_p r_p^2}, \tag{3}$$

where as before $v=\|\mathbf{v}\|$ and $u=\|\mathbf{u}\|$. Equation (3) gives, implicitly, the norm of the average particle velocity u as a function of the norm of the macroscopic fluid velocity v. A plot of u/α versus v/α is shown in Fig. 2.

The main properties of u as a function of v are (1) $0<u<v$ for all $v>0$, (2) u is an increasing function of v, (3) $u\approx v$ for $v\ll\alpha$, and (4) $u\approx\sqrt{\alpha v/2}$ for $v\gg\alpha$. As an example, we will now use Eq. (3) to model particle transport and possible clogging in a two-dimensional model of a well production. Assume that our two-dimensional porous medium occupies the region of the plane $\Omega=\{R_i\leq\|\mathbf{x}\|\}$, where R_i is a constant. Note that $\{\|\mathbf{x}\|\leq R_i\}$ is the space occupied by the well. We will denote by $\phi=\phi(\mathbf{x})$ the porosity, i.e., the local volume fraction of void or pore space. Note that $\phi(\mathbf{x})$ is independent of time. We denote by $z=z(\mathbf{x},t)$ the volume fraction of fine particles. Note that the volume fraction of fluid is $\phi(x)-z(\mathbf{x},t)$. We refer to z as the concentration of particles. Since we assume that both particles and fluid are incompressible, the equation of mass conservation becomes $\nabla\cdot[\mathbf{v}(\phi-z)+\mathbf{u}z]=0$, where $\nabla\cdot$ is the divergence operator. We assume the small concentration of particle limit, i.e., $z\ll\phi$, and that ϕ is independent of $\mathbf{x}$. Thus, mass conservation reduces to $\nabla\cdot\mathbf{v}=0$. We also assume that fluid flows at a known constant rate and with circular symmetry toward the inner boundary $\{\|\mathbf{x}\|=R_i\}$. Thus, introducing the radial variable $r=\|\mathbf{x}\|$ we have that $\nabla\cdot\mathbf{v}=0$ implies that the velocity $\mathbf{v}$ is of the form $\mathbf{v}=-v\mathbf{x}/r$ with $v=A/r$, where A is a known positive constant. Note that $2\pi\phi A$ is the rate at which fluid exits the medium through the inner boundary (the well).

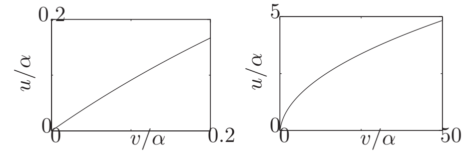

FIG. 2. Plot of u/α vs v/α. Both plots correspond to the same curve. They are just in different scales.

The macroscopic fine particle velocity will also have a similar form as the macroscopic fluid velocity $\mathbf{v}$, i.e.,

$$\mathbf{u}=-u\frac{\mathbf{x}}{r}, \tag{4}$$

where u and v are related by Eq. (3) and thus $u=u(r)$. Since we now have v in terms of r, Eq. (3) gives us a relation between u and r, namely,

$$\frac{\alpha r}{A}=\frac{\alpha}{u}-(1-e^{-\alpha/u}), \tag{5}$$

where α was defined in Eq. (3). The main properties of u as a function of r are as follows: (1) u is a decreasing function of r, (2) $u\approx A/r$ for $r\gg A/\alpha$, and (3) $u\approx\sqrt{\alpha A/(2r)}$ for $r\ll A/\alpha$.

Since the concentration of fine particles z is convected with the macroscopic velocity of fine particles $\mathbf{u}$, and given the circular symmetry of our problem, we have the following conservation equation:

$$\frac{\partial z}{\partial t}-\frac{1}{r}\frac{\partial(ruz)}{\partial r}=0, \tag{6}$$

where $u=u(r)$ is a function of r given implicitly in Eq. (5). We remark that the dependence of $u=u(r)$ as a function of r was obtained under the assumption that z is small, i.e., $z\ll\phi$. Nevertheless, we will extend the use of Eq. (5) beyond the restriction $z\ll\phi$ to add clarity to our exposition while keeping the physical effects we are interested in modeling. Note that the evolution of the concentration of fine particles $z(r,t)$ is completely determined by Eqs. (5) and (6) once the initial conditions are specified, i.e., we need to know $z(r,0)$.

In Fig. 3 we show an example of the time evolution of the concentration of fine particles z. We plotted z versus r/R_i

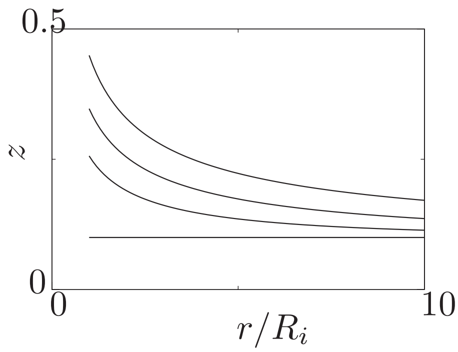

FIG. 3. Plot of z vs r/R_i for four fixed values of t: $t=0$, $t=R_i/\alpha$, $t=5R_i/\alpha$, and $t=+\infty$. z increases with t but remains bounded.

for four fixed values of t: $t=0$, $t=R_i/\alpha$, $t=5R_i/\alpha$, and $t=+\infty$. The initial condition was $z(r,0)=0.1$. Note that $z(r,t)$ is defined only for $r \geqslant R_i$.

In the rest of this letter we assume that the initial conditions are homogeneous, i.e., $z(r,0)=z_0$ for all r, where z_0 is a constant. Our model predicts that the concentration of particles will increase with time, but it will remain bounded. The concentration will not remain homogeneous. For any positive time, the concentration of particles is a decreasing function of the distance to the well.

It is of interest to predict if clogging occurs. Let us now introduce a simple criterion for clogging in our modeling context. Regions of the medium clog if and when the concentrations of fine particles z in those regions exceed a certain critical value $z_\star$. Thus, in our example, if clogging occurs, it will happen at the boundary between the medium and the well, i.e., at $r=R_i$. Once this ring is clogged, there is no more flow through the medium.

Clogging may or may not occur. The outcome will depend on the parameters of the system. In fact, solving our model shows the following.

Observation 1. Let s_ℓ be the root of $\alpha R_i/A=s_\ell^{-1}-(1-e^{-1/s_\ell})$. The medium clogs if and only if $1-s_\ell(1-e^{-1/s_\ell})<z_0/z_*$.

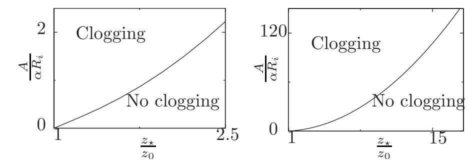

FIG. 4. Regions in the parameter plane $A/(\alpha R_i)$ vs z_*/z_0 where clogging does and does not occur. Both plots correspond to the same regions. They are just in different scales.

In Fig. 4 we display the regions in the parameter plane $A/(\alpha R_i)$ versus z_*/z_0 where clogging does and does not occur. The parameter z_*/z_0 determines how many times the concentration of fine particles needs to increase for the medium to plug. Note that A/R_i is the macroscopic fluid velocity v at the inner boundary and α, defined in Eq. (3), is a velocity that depends on microscopic or pore-scale parameters of the system. Note that α indicates when the difference between the macroscopic fluid velocity $\mathbf{v}$ and macroscopic velocity of fine particles $\mathbf{u}$ is noticeable. More precisely, $u \approx v$ if and only if $v \ll \alpha$.

In summary, we have developed a simple mathematical model that provides a relation between the average fluid velocity and the average velocity of the transported particles, in the parameter regime of relative large fluid velocities and particles size. We have shown the parameter regime where these physical effects are important. We have also shown that these effects may lead to the clogging of porous media if the fluid velocity field is not homogeneous, which is the case of velocity fields near production wells.

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