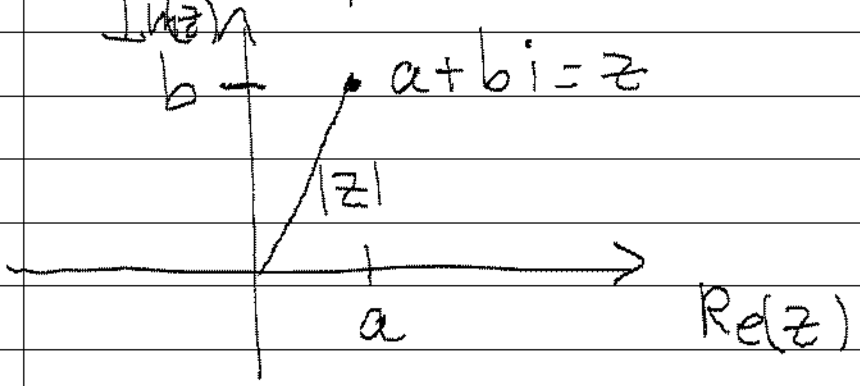
Note Title 6/3/2015

<u>Def</u>: $z = a+bi$ a and $b \in \mathbb{R}$. The complex conjugate of z is $\bar{z} = a - bi$

<u>Example</u>: $z = 1+2i$ then $\bar{z} = 1-2i$

<u>Obs</u>: $z\bar{z} = (a+bi)(a-bi) = a^2 - (bi)^2 = a^2+b^2$

<u>Complex plane</u>



<u>Def</u>: The modulus or absolute value of $z = a+bi$ is $|z| = \sqrt{a^2+b^2} = \sqrt{z\bar{z}}$

Def: $\mathbb{C}^n$ is the set of all n-vectors whose components are complex numbers $z \in \mathbb{C}^n$, then

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \qquad z_i \in \mathbb{C}$$

Obs: $z = a + bi \qquad w = c + di$

$$\frac{z}{w} = \frac{z\,\overline{w}}{w\,\overline{w}} = \frac{z\overline{w}}{|w|^2} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)}{(c^2+d^2)} + \frac{(bc-ad)\,i}{(c^2+d^2)}$$

Obs: $A \in \mathbb{R}^{n\times n}$ we can have complex eigenvalues and eigenvectors

$$Av = \lambda v \qquad \lambda \in \mathbb{C} \text{ and } v \in \mathbb{C}^n \qquad v \neq 0$$

Obs: Let $P(x)$ be a real polynomial. Let $z \in \mathbb{C}$. If $P(z) = 0$ then

(*) $P(z) = \boxed{a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z_1 + a_0 = 0}$

(note: If w_1 and w_2 are two complex numbers, then $\overline{w_1 + w_2} = \overline{w_1} + \overline{w_2}$ and $\overline{w_1 w_2} = \overline{w_1}\,\overline{w_2}$)

Back to (*). Take complex conjugate of (*)

$$\overline{a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0} = \overline{0}$$

then $\overline{a_n}\,\overline{z}^n + \overline{a_{n-1}}\,\overline{z}^{n-1} + \dots + \overline{a_1}\,\overline{z} + \overline{a_0} = 0$ but $a_i \in \mathbb{R}$

$$a_n \overline{z}^n + a_{n-1} \overline{z}^{n-1} + \dots + a_1 \overline{z} + a_0 = 0 \quad , \quad \text{i.e.}$$

$$\boxed{P(\bar{z}) = 0}$$

We proved that, if $P(x)$ is real, then $P(z)=0$ if and only if $P(\bar{z})=0$

Example: both $-1+i$ and $-1-i$ are roots of $x^2+2x+2=0$

Obs: Let $A \in \mathbb{R}^{n\times n}$. Then $\boxed{Av = \lambda v \text{ if and only if } A\bar{v} = \bar{\lambda}\bar{v}}$

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \text{ then } \bar{v} = \begin{bmatrix} \bar{v}_1 \\ \vdots \\ \bar{v}_n \end{bmatrix}$$

λ is an eigenvalue of A with eigenvector v if and only if $\overline{\lambda}$ is an eigenvalue of A with eigenvector $\overline{v}$ $(A \in \mathbb{R}^{n\times n})$

<u>Example</u>: Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$$

$$P(\lambda) = (6-\lambda)(4-\lambda) + 5 = \lambda^2 - 10\lambda + 29 = 0$$

$$\boxed{\lambda} = \frac{10 \pm \sqrt{100-116}}{2} = \frac{10 \pm \sqrt{-16}}{2} = \frac{10 \pm i\sqrt{16}}{2} = \boxed{5 \pm 2i}$$

Eingenvector of $\boxed{\lambda_1 = 5+2i}$

$$A-\lambda_1 I = \begin{bmatrix} 1-2i & -1 \\ 5 & -1-2i \end{bmatrix} \qquad (A-\lambda_1 I)v = 0$$

$(1-2i)\,v_1 + (-1)\,v_2 = 0$ to get a solution set $v_1 = 1$

$$\boxed{v = \begin{bmatrix} 1 \\ 1-2i \end{bmatrix}}$$

Eigenvectors	Eigenvalues
$\begin{bmatrix} 1 \\ 1-2i \end{bmatrix}$	$5+2i$
$\begin{bmatrix} 1 \\ 1+2i \end{bmatrix}$	$5-2i$

Obs: $\lambda = 0$ is an eigenvalue of A if and only if A is singular

Obs: $\lambda \neq 0$, A non-singular, $Av = \lambda v$ $\quad v \neq 0$.

$$A^{-1}(Av) = A^{-1}(\lambda v)$$
$$(A^{-1}A)v = \lambda A^{-1} v$$
$$Iv = \lambda A^{-1} v$$
$$v = \lambda A^{-1} v$$
$$\boxed{\lambda^{-1} v = A^{-1} v}$$

A non-singular. λ is an eigenvalue of A with eigenvector v if and only if λ^{-1} is an eigenvalue of A^{-1} with eigenvector v.

Symmetric matrices

__Def:__ A is symmetric if $A^T = A$

__Example__ $\begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}$ is symmetric

__Obs:__ $Av = \lambda v$ $A \in \mathbb{R}^{n\times n}$ $\boxed{A \text{ symmetric}}$ $\lambda \in \mathbb{C}$ $v \in \mathbb{C}^n$

$\bar{v}^T(Av) = \bar{v}^T(\lambda v)$

$\boxed{\bar{v}^T A v = \lambda\, \bar{v}^T v}$

transpose

$(\bar{v}^T A v)^T = \lambda(\bar{v}^T v)^T$

$v^T A^T (\bar{v}^T)^T = \lambda\, v^T(\bar{v}^T)^T$

then $v^T A \bar{v} = \lambda(v^T \bar{v})$

take complex conjugate

$\bar{v}^T \bar{A} v = \bar{\lambda}(\bar{v}^T v)$ but A is real

$\boxed{\bar{v}^T A v = \bar{\lambda}(\bar{v}^T v)}$ then

$\boxed{\lambda(\bar{v}^T v) = \bar{\lambda}(\bar{v}^T v)}$ $(*)$

$$\bar{v}^T v = [\bar{v}_1 \ \bar{v}_2 \ \dots \ \bar{v}_n] \begin{bmatrix} v_1 \\ v_2 \\ \\ v_n \end{bmatrix} = |v_1|^2 + |v_2|^2 + \dots + |v_n|^2 > 0$$

because $v \neq 0$

then (*) implies $\boxed{\lambda = \bar{\lambda}}$

Theorem: all the eigenvalues of a symmetric real matrix are real.

Inner product: $x, y \in \mathbb{R}^n$

$$x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = [x_1 \ x_2 \ \dots \ x_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x^T y$$

$$\|x\| = \sqrt{x \cdot x} = \sqrt{x^T x}$$

x and y are orthogonal if $x^T y = 0$

Theorem: $A \in \mathbb{R}^{n \times n}$. A symmetric. Eigenvectors of A of different eigenvalues are orthogonal.

proof: $A v_1 = \lambda_1 v_1$ and $A v_2 = \lambda_2 v_2$ $\quad \lambda_1 \neq \lambda_2 \quad v_1 \neq 0$, $v_2 \neq 0$

$$v_2^T A v_1 = \lambda_1 v_2^T v_1 \qquad \boxed{v_1^T A v_2 = \lambda_2 v_1^T v_2}$$

transpose

$$v_1^T A^T v_2 = \lambda_1 v_1^T v_2$$

but $A^T = A$

$$\boxed{v_1^T A v_2 = \lambda_1 v_1^T v_2} \longrightarrow \lambda_1 v_1^T v_2 = \lambda_2 v_1^T v_2$$

but $\lambda_1 \neq \lambda_2 \Longrightarrow \boxed{v_1^T v_2 = 0}$

Obs: $A \in \mathbb{R}^{n\times n}$. $P(\lambda) = \det(A - \lambda I)$

$P(\lambda)$ is a polynomial of degree n. There exists $\lambda_1, \dots, \lambda_r \in \mathbb{C}$ and $n_1, \dots, n_r$ positive integers such that

$$\boxed{\begin{array}{c} n_1 + n_2 + \cdots + n_r = n \quad \text{and} \\ P(\lambda) = (-1)^n (\lambda - \lambda_1)^{n_1} \cdots\cdots (\lambda - \lambda_r)^{n_r} \\ \lambda_i \neq \lambda_j \quad \text{if} \quad i \neq j \end{array}}$$

$\lambda_1, \lambda_2, \dots, \lambda_r$ are the different eigenvalues

n_i is called the algebraic multiplicity of λ_i

The geometric multiplicity of λ_i is the number of linearly independent eigenvectors of A with eigenvalue λ_i = number of free variables in the system $(A-\lambda_i I)x=0$ (same as number of parameters $t_1, t_2, \ldots$ in the solution).

<u>Obs</u>: $\boxed{\text{geometric multiplicity of } \lambda_i \leq \text{algebraic multiplicity of } \lambda_i}$

<u>Obs</u>: If A is symmetric,

$\boxed{\text{geometric multiplicity of } \lambda_i = \text{algebraic multiplicity of } \lambda_i}$

<u>Theorem</u>: If A is symmetric, then A has n linearly independent eigenvectors.

Orthogonal matrices: A is orthogonal if $A^{-1} = A^T$, i.e. $A \in \mathbb{R}^{n \times n}$ and $A^T A = I$

Example: $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ is orthogonal

Check

$$A^T A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Obs: $B = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}$ $\quad A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$ $\quad a_i \in \mathbb{R}^n$

$b_i \in \mathbb{R}^n$

$A, B \in \mathbb{R}^{n \times n}$

$$A^T B = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_n \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_n \\ \vdots & \vdots & & \vdots \\ a_n^T b_1 & a_n^T b_2 & \cdots & a_n^T b_n \end{bmatrix}$$

$C = A^T B \qquad c_{ij} = a_i \cdot b_j$

<u>Obs</u>: $A \in \mathbb{R}^{n \times n}$ $\quad B = A^T A \quad$ b_{ij} = i^{th} column of A inner product j^{th} column of A

<u>Theorem</u>: $A \in \mathbb{R}^{n \times n}$. A is orthogonal if and only if

i^{th} column of A inner product j^{th} column of A $= \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

$A = [a_1\ a_2 \dots a_n]$ $\quad a_i = i^{th}$ column of A

A is orthogonal if and only if the set $a_1, a_2, \dots, a_n$ is orthonormal.

Diagonalization

Def: A matrix A is diagonalizable if $A = PDP^{-1}$ for some diagonal matrix D. A, D and $P \in \mathbb{R}^{n \times n}$

Fact: $P \in \mathbb{R}^{n \times n}$ has an inverse if and only if the columns of P are linearly independent.

Obs: $A = PDP^{-1}$ then $AP = PD$

$P = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix}$ p_j is the j^{th} column of P

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

$$\boxed{AP = A\begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix}} = \boxed{\begin{bmatrix} Ap_1 & Ap_2 & \cdots & Ap_n \end{bmatrix}}$$

$$PD = P\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix} = \begin{bmatrix} P\begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & P\begin{bmatrix} 0 \\ \lambda_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \cdots & P\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_n \end{bmatrix} \end{bmatrix}$$

j^{th} column of $AP = A p_j$

j^{th} column of $PD = P \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_j \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_j p_j$

$$\boxed{A p_i = \lambda_i p_i}$$

<u>Obs</u>: A is diagonalizable if and only if A has n linearly independent eigenvectors.