Note Title 5/18/2015

Subspaces

Def: Let V be a vector space. W is a subspace of V if $W \subseteq V$ and W is a vector space

Example: $P_n = \{$ polynomials of degree n or less and the polynomial $0 \}$

P_n is a subspace of P, the set of all polynomials

Theorem: Let V be a vector space. Let $W \subseteq V$. W is a subspace if and only if:

1) $0 \in W$

2) $x, y \in W$ implies $x+y \in W$

3) $x \in W$ and $\lambda \in \mathbb{R}$ then $\lambda x \in W$

Def: Let $v_1, v_2, \dots, v_r \in V$. Let $x \in V$. We say that x is a linear combination of $v_1, v_2, \dots, v_r$ if there exists $a_1, a_2, \dots, a_r \in \mathbb{R}$ such that

$$x = a_1 v_1 + a_2 v_2 + \dots + a_r v_r$$

Example: $(-1,0) = 2(1,1)+(-1)(3,2)$. Thus, $(-1,0)$ is a linear combination $(1,1)$ and $(3,2)$.

Def: We say that a set $\{v_1, v_2, \dots, v_r\} \subset \mathbb{R}^n$ is orthogonal if $v_i \cdot v_j = 0$ if $i \neq j$.

Def: We say that a set $v_1, v_2, \dots, v_r$ is orthonormal if it is orthogonal and $\|v_i\| = 1$ for all $1 \le i \le r$

<u>Example</u>: $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), (0,0,1)$ is orthonormal.

<u>Def</u>: Let $v_1, v_2, \dots, v_r \in V$

$\operatorname{span}\{v_1, v_2, \dots, v_r\} = \{$ linear combinations of $v_1, v_2, \dots, v_r\}$

<u>Examples</u> 1) $\operatorname{span}\{(1,0),(0,1)\} = \mathbb{R}^2$ because for any $x \in \mathbb{R}^2$ we have $x = (x_1, x_2) = x_1(1,0) + x_2(0,1)$

2) $\operatorname{span}\{(1,1,0),(1,-1,0)\} = \{x \in \mathbb{R}^3 : x_3 = 0\}$ because, if $x_3 = 0$ we have $x = (x_1, x_2, 0) = \frac{(x_1+x_2)}{2}(1,1,0) + \frac{(x_1-x_2)}{2}(1,-1,0)$

3) $\text{span}\{1, x, x^2\} = P_2 = \{$ polynomials of degree 2 or less and the polynomial $0\}$ because, if $p \in P_2$, then $p = a_2 x^2 + a_1 x + a_0$ for some $a_0, a_1, a_2 \in \mathbb{R}$.

Obs: $\text{span}\{v_1, v_2, \dots, v_r\}$ is a subspace

proof: 1) Is $0 \in \text{span}\{v_1, v_2, \dots, v_r\}$? Yes, because $0 = 0v_1 + 0v_2 + \dots + 0v_r$

2) Let $x, y \in \text{span}\{v_1, v_2, \dots, v_r\}$. Then, there exists $a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_r \in \mathbb{R}$ such that

$$x = a_1 v_1 + a_2 v_2 + \dots + a_r v_r \quad \text{and}$$

$$y = b_1 v_1 + b_2 v_2 + \dots + b_r v_r$$

then $x+y=(a_1+b_1)v_1+(a_2+b_2)v_2+\dots+(a_r+b_r)v_r$ then

$x+y \in \operatorname{span}\{v_1, v_2, \dots, v_r\}$

3) $x \in \operatorname{span}\{v_1, v_2, \dots, v_r\}$, $\lambda \in \mathbb{R}$, then $\lambda x \in \operatorname{span}\{v_1, v_2, \dots, v_r\}$

(you check)

<u>Def</u>: We say that a set of vectors $v_1, v_2, \dots, v_r$ is linearly independent if $0=c_1v_1+c_2v_2+\dots+c_rv_r$ implies $c_1=c_2=\dots=c_r=0$

<u>Examples</u> 1) $(1,0,0), (0,1,0), (0,0,1)$ is linearly independent.

<u>proof</u>: $c_1(1,0,0)+c_2(0,1,0)+c_3(0,0,1)=(0,0,0)$

then $(c_1, c_2, c_3)=(0,0,0)$ then $c_1=c_2=c_3=0$

2) $(1,0), (0,1), (2,1)$ is not linearly independent because

$(0,0) = 2(1,0) + 1(0,1) + (-1)(2,1)$

<u>Def</u>: linearly dependent = not linearly independent

<u>Question</u>: If $x \in \operatorname{span}\{v_1, v_2, \dots, v_r\}$. $x, v_1, v_2, \dots, v_r$ given.

How do we find $a_1, a_2, \dots, a_r$ such that $x = a_1 v_1 + a_2 v_2 + \dots + a_r v_r$?

<u>Partial answer</u>: If $v_1, v_2, \dots, v_r$ is orthogonal, then

$$x = a_1 v_1 + a_2 v_2 + \dots + a_r v_r$$

take dot product with v_i

$$x \cdot v_i = a_1(v_1 \cdot v_i) + a_2(v_2 \cdot v_i) + \dots \dots + a_r(v_r \cdot v_i)$$

$v_i \cdot v_j = 0$ if $i \neq j$ then

$$X \cdot v_i = a_i (v_i \cdot v_i)$$

then $\boxed{a_i = \frac{X \cdot v_i}{v_i \cdot v_i}}$ assuming $v_i \neq 0$

In particular, if $v_1, v_2, \ldots, v_r$ is orthonormal

$$\boxed{a_i = X \cdot v_i}$$

<u>Example</u>: $X = (x_1, x_2, 0)$ $\quad v_1 = (1,1,0)$ and $v_2 = (1,-1,0)$

note that $v_1 \cdot v_2 = 0$ thus v_1, v_2 is orthogonal

$$a_1 = \frac{X \cdot v_1}{v_1 \cdot v_1} = \frac{(x_1, x_2, 0) \cdot (1,1,0)}{(1,1,0) \cdot (1,1,0)} = \frac{x_1 + x_2}{2}$$

$$a_2 = \frac{X \cdot v_2}{v_2 \cdot v_2} = \frac{x_1 - x_2}{2}$$

Def: Let W be a subspace of V. Let $v_1, v_2, \dots, v_r \in V$. We say that $v_1, v_2, \dots, v_r$ is a basis of W if

1) $W = \text{span}\{v_1, v_2, \dots, v_r\}$

2) $v_1, v_2, \dots, v_r$ is linearly independent

Example: 1) $(1,0), (0,1)$ is a basis of $\mathbb{R}^2$

2) $1, x, x^2$ is a basis for P_2

check: $P_2 = \text{span}\{1, x, x^2\}$ obvious ✓

$0 = a(1) + bx + cx^2$

Set $x=0$ to get $0=a(1)=a$ $\boxed{a=0}$

Set $x=1$ to get $0=b+c$ } add to get $0=2c$ $\boxed{c=0}$

Set $x=-1$ to get $0=-b+c$ } substract to get $\boxed{b=0}$

then $1, x, x^2$ is linearly independent.

Def: $v_1, v_2, \dots, v_r$ is an orthogonal (orthonormal) basis of W if it is a basis for W and it is an orthogonal (orthonormal) set.

Gram-Schmidt: $v_1, v_2, \dots, v_r$ linearly independent. Given

Goal: Find $u_1, u_2, \dots, u_r$ orthonormal such that

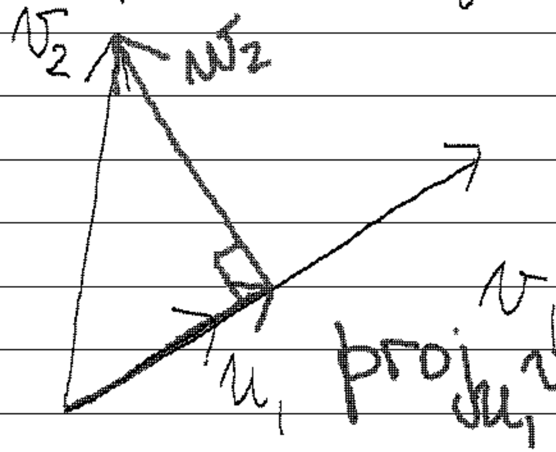
$\text{span}\{u_1, u_2, \dots, u_t\} = \text{span}\{v_1, v_2, \dots, v_t\}$ for all $1 \le t \le r$

1) $\mathrm{span}\{u_1\} = \mathrm{span}\{v_1\}$ implies $u_1 \in \mathrm{span}\{v_1\} \Longrightarrow u_1 = \lambda v_1$ for some $\lambda \in \mathbb{R}$. $1 = \|u_1\| = |\lambda| \, \|v_1\|$ then $\lambda = \dfrac{1}{\|v_1\|}$

$$\boxed{u_1 = \frac{v_1}{\|v_1\|}}$$

2) $\mathrm{span}\{u_1, u_2\} = \mathrm{span}\{v_1, v_2\} = \mathrm{span}\{u_1, v_2\}$

$u_2 \in \mathrm{span}\{u_1, v_2\}$



$\mathrm{proj}_{u_1} v_2 = (u_1 \cdot v_2) u_1$

$$\boxed{w_2 = v_2 - \mathrm{proj}_{u_1} v_2 = v_2 - (u_1 \cdot v_2)\, u_1}$$

$$\boxed{u_2 = \frac{w_2}{\|w_2\|}}$$

3) $$\boxed{w_i = v_i - (u_1 \cdot v_i)\, u_1 - (u_2 \cdot v_i)\, u_2 - \ldots\ldots - (u_{i-1} \cdot v_i)\, u_i}$$

$$\boxed{u_i = \frac{w_i}{\|w_i\|}} \qquad i = 1, 2, \ldots, r$$

Example $v_1 = (1,1,1)$ $\quad v_2 = (1,2,2)$ $\quad v_3 = (1,1,0)$

$$\boxed{u_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$$

$$w_2 = v_2 - (u_1 \cdot v_2)\, u_1 = (1,2,2) - \left(\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \cdot (1,2,2)\right)\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) =$$

$$= \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$\boxed{u_2 = \frac{w_2}{\|w_2\|} = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)}$$

$$w_3 = v_3 - (u_1 \cdot v_3)\, u_1 - (u_2 \cdot v_3)\, u_2 = (1,1,0) - \left[\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \cdot (1,1,0)\right] \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$- \left[\left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \cdot (1,1,0)\right] \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = \left(0, \frac{1}{2}, -\frac{1}{2}\right)$$

$$\boxed{u_3 = \frac{w_3}{\|w_3\|} = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)}$$