f analytic inside $\mathscr{C}$, then $\oint_{\mathscr{C}} f = 0$

<u>Obs</u>: f is analytic inside the region bounded by $\mathscr{C}_1$ & $\mathscr{C}_2$. Then

$$0 = \oint_{\mathscr{C}_1 \cup -\mathscr{C}_2} f = \int_{\mathscr{C}_1} f - \int_{\mathscr{C}_2} f \qquad \text{then}$$

$$\boxed{\int_{\mathscr{C}_1} f(z)\,dz = \int_{\mathscr{C}_2} f(z)\,dz}$$

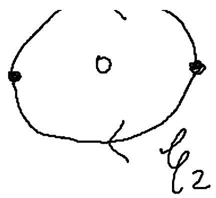
<u>Theorem</u>: D simply connected. f analytic in D. $\mathscr{C}$ curve in D. Then $\int_{\mathscr{C}} f(z)\,dz$ depends only on the initial and final points

$$\int_{\bar{\mathscr{C}}} f(z)\,dz = \int_{\mathscr{C}} f(z)\,dz$$

<u>Example</u>: $f(z) = \frac{1}{z}$ $\qquad \mathscr{C}_1 : z = e^{it} \quad 0 \le t \le \pi$

$\qquad \mathscr{C}_2 : z = e^{-it} \quad 0 \le t \le \pi$

$$\int \frac{dz}{} = \int^{\pi} \frac{i e^{it}}{} dt = \pi i$$



$$\int_{\mathcal{C}_1} \frac{dz}{z} = \int_0^{\pi} \frac{i e^{it}}{e^{it}}\, dt = \pi i$$

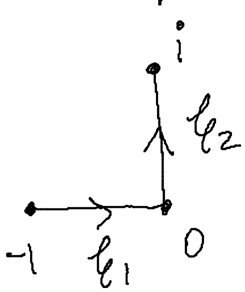
$$\int_{\mathcal{C}_2} \frac{dz}{z} = \int_0^{\pi} \frac{-i e^{it}}{e^{-it}}\, dt = -\pi i$$

$\int_{\mathcal{C}_1} \frac{dz}{z} \neq \int_{\mathcal{C}_2} \frac{dz}{z}$. this does not contradict the theorem because $\frac{1}{z}$ is not analytic everywhere inside the region bounded by $\mathcal{C}_1 \cup \mathcal{C}_2$.

<u>Theorem</u>: D simply connected. f analytic in D. $\mathcal{C}$ curve inside D. z_0 & z_1 the initial and final points of $\mathcal{C}$. Assume $F'(z) = f(z)$ for all $z \in D$.

then $$\boxed{\int_{\mathcal{C}} f(z)\,dz = F(z_1) - F(z_0)}$$

<u>Example</u>: $f(z) = z^2$ $\qquad \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$



Without using the theorem

$$\int_{\mathcal{C}} z^2\,dz = \int_{\mathcal{C}_1} z^2\,dz + \int_{\mathcal{C}_2} z^2\,dz =$$

$$= \int_{-1}^{0} t^2\,dt + \int_0^1 (it)^2\, i\,dt = \left.\frac{t^3}{3}\right|_{-1}^{0} - i \left.\frac{t^3}{3}\right|_0^1 =$$

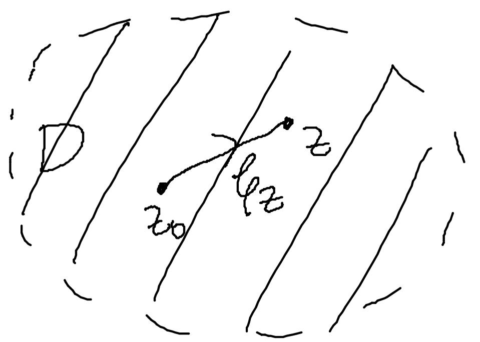
$$= \frac{1}{3} - i\frac{1}{3} = \frac{1-i}{3}$$

$\mathcal{C}_1 : z = t \quad -1 \le t \le 0$ $\qquad = \frac{1}{3} - i\frac{1}{3} = \frac{1-i}{3}$

$\mathcal{C}_2 : z = it \quad 0 \le t \le 1$ $\quad$ Using the theorem

$$\int_{\mathcal{C}} z^2\,dz = \frac{z^3}{3}\Bigg|_{-1}^{i} = \frac{i^3}{3} - \frac{(-1)^3}{3} = \frac{1-i}{3}$$

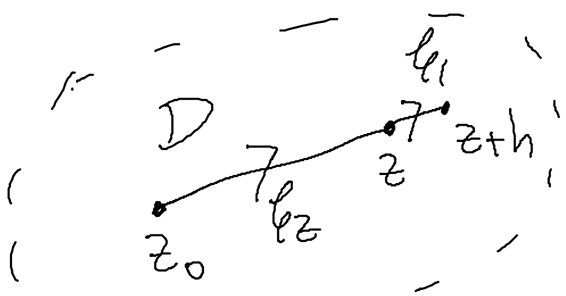
<u>Theorem</u>: D simply connected. f analytic in D. Then, there exist F in D such that $F'(z) = f(z)$ for all $z \in D$



$z_0 \in$ fixed. $z \in D$

$\mathcal{C}_z$ a curve that connects z_0 with z and $\mathcal{C}_z \subset D$. Define

$$F(z) = \int_{\mathcal{C}_z} f(z)\,dz$$



$$F(z+h) = \int_{\mathcal{C}_z \cup \mathcal{C}_1} f(z)\,dz$$

$$-F(z) = -\int_{\mathcal{C}_z} f(z)\,dz$$

<u>Cauchy's integral formula</u>

D simply connected. $\mathcal{C}$ closed curve.

$\mathcal{C} \subset D$. $\mathcal{C}$ does not self-intersect

$\mathcal{C} \subset D$. $\mathcal{C}$ does not self-intersect.

z_0 inside $\mathcal{C}$. f is analytic in D.

then $$\boxed{f(z_0) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{z - z_0}\, dz}$$

__proof__ $g(z) = \dfrac{f(z)}{z - z_0}$ is analytic in $D - \{z_0\}$

From last class,

$$\boxed{\oint_{\mathcal{C}} g(z)\, dz = \lim_{\varepsilon \to 0} \int_{|z - z_0| = \varepsilon} g(z)\, dz} =$$

$$= \lim_{\varepsilon \to 0} \int_0^{2\pi} g(z_0 + \varepsilon e^{it})\, \varepsilon i e^{it}\, dt =$$

$$\boxed{\begin{array}{l} |z - z_0| = \varepsilon \\ z = z_0 + \varepsilon e^{it} \\ 0 \le t \le 2\pi \end{array}}$$

$$= \lim_{\varepsilon \to 0} \int_0^{2\pi} \frac{f(z_0 + \varepsilon e^{it})}{\cancel{\varepsilon} \cancel{e^{it}}}\, \cancel{\varepsilon} i \cancel{e^{it}}\, dt =$$

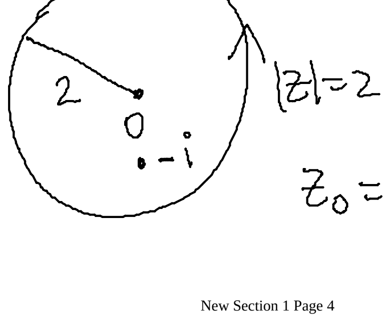
$$\boxed{= 2\pi i\, f(z_0)}$$

__Example__: $$\boxed{\oint_{|z|=2} \frac{z^2 - 4z + 4}{z + i}\, dz} = \oint_{|z|=2} \frac{f(z)}{z - (-i)}\, dz$$

$$= \oint_{|z|=2} \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0)$$

$$= 2\pi i\, (-1 + 4i + 4)$$

$$\boxed{= 2\pi i\, (3 + 4i)}$$



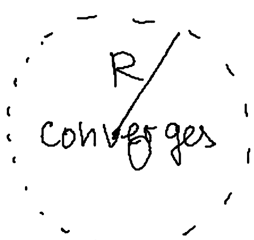
$z_0 = -i$ inside $\mathcal{C}$: $|z| = 2$

$$f(z) = z^2 - 4z + 4$$

Power series: $\displaystyle\sum_{k=0}^{\infty} a_k z^k = \lim_{N\to\infty} \sum_{k=0}^{N} a_k z^k$

1) there exists $R \geq 0$ (R could be ∞) such that the above limit exists for all $|z| < R$, but it does not exist for all $|z| > R$. R is called the radius of convergence

diverges



2) $f(z) = \displaystyle\sum_{k=0}^{\infty} a_k z^k$. $f(z)$ is analytic for $|z| < R$

3) If $R > 0$, $a_k = \dfrac{f^{(k)}(0)}{k!}$ (Taylor).

4) If $R > 0$, then $f'(z) = \displaystyle\sum_{k=0}^{\infty} k a_k z^{k-1}$

the radius of convergence of this series is also R.

5) If $R > 0$, $F(z) = \displaystyle\sum_{k=0}^{\infty} \frac{a_k}{k+1} z^{k+1}$. This series also has radius of convergence R and $F'(z) = f(z)$

6) 

f analytic in D. $\{|z-z_0|<r\} \subset D$

then $$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

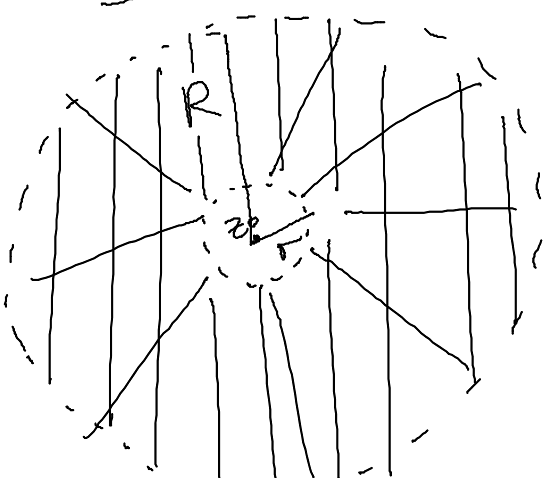
for all $|z-z_0|<r$, i.e. the radius of convergence of is $\geq r$. this is called the Taylor series of f at z_0.

<u>Examples</u>: 1) $$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad \text{for all } z \in \mathbb{C}.$$

2) $$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \quad \text{for all } |z|<1$$

<u>Laurent series</u>

$\{r<|z-z_0|<R\} \subset D$



f analytic in D.

then f can be expanded as a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$ for all z such that $r<|z_0-z|<R$

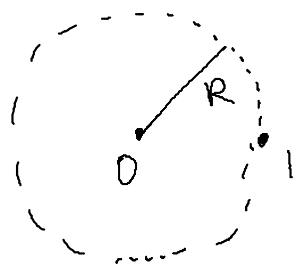
<u>Examples</u>: 1) $f(z) = \frac{1}{z}$ $\quad z_0=0 \quad r=0 \quad R=\infty$

$a_{-1}=1 \quad a_k=0$ for all $k \neq -1$

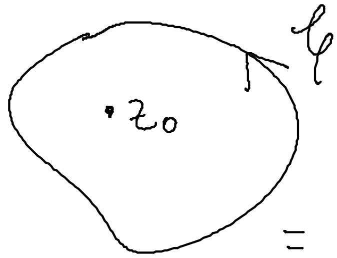
2) $f(z) = \dfrac{1}{z(1-z)}$. Find the Laurent seies of f at $z_0 = 0$

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \qquad \frac{1}{z(1-z)} = \sum_{k=0}^{\infty} \frac{z^k}{z} = \sum_{n=-1}^{\infty} z^n$$

$r = 0 \quad R = 1$



<u>Obs</u>: z_0 inside $\mathcal{C}$. $\mathcal{C}$ closed curve



$$\oint_{\mathcal{C}} (z-z_0)^k \, dz = \lim_{\varepsilon \to 0} \oint_{|z-z_0| = \varepsilon} (z-z_0)^k \, dz =$$

$$= \lim_{\varepsilon \to 0} \int_0^{2\pi} (\varepsilon e^{it})^k \varepsilon i e^{it} \, dt =$$

$$= \lim_{\varepsilon \to 0} i \varepsilon^{k+1} \int_0^{2\pi} e^{it(k+1)} \, dt = \begin{cases} 2\pi i & \text{if } k = -1 \\ \lim\limits_{\varepsilon \to 0} i \varepsilon^{k+1} \left. \dfrac{e^{it(k+1)}}{i(k+1)} \right|_0^{2\pi} = 0 \end{cases}$$

$$\oint_{\mathcal{C}} (z-z_0)^k \, dz = \begin{cases} 0 & \text{if } k \neq -1 \\ 2\pi i & \text{if } k = -1 \end{cases}$$

<u>Def</u>: $f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$

$a_{-1} = \mathrm{Res}(f, z_0)$ is the residue of f at z_0

<u>Example</u> 1) $f(z) = \dfrac{1}{\ldots} = \dfrac{1}{z} + 1 + z + z^2 + \ldots$

Example 1) $f(z) = \frac{1}{(1-z)\,z} = \frac{1}{z} + 1 + z + z^2 + \cdots$

$$\boxed{\operatorname{Res}\left(\frac{1}{(1-z)\,z}, 0\right) = 1}$$

2) $f(z) = \frac{1}{(1-z)\,z}$ at $z_0 = 1$

$$\frac{(-1)}{(z-1)(z-1+1)} = \frac{(-1)}{(z-1)}\,\frac{1}{\left[1-(-1)(z-1)\right]} = \frac{-1}{z-1}\sum_{k=0}^{\infty}(-1)^k(z-1)^k =$$

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n \qquad = \frac{(-1)}{(z-1)} + 1 - (z-1) + (z-1)^2 - \cdots$$

$$\boxed{\operatorname{Res}\left(\frac{1}{(1-z)\,z}, 1\right) = -1}$$