Homogeneous systems of 1st order linear constant coefficient equations

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n \\
&\vdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{nn}x_n
\end{aligned}$$

$a_{ij} \in \mathbb{R}$

a_{ij} known

Goal: find $x_i(t)$ $1 \le i \le n$

Matrix notation

$$x = x(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \boxed{\dot{x} = Ax}$$

Obs: Let $x = e^{\lambda t}v$, $v \in \mathbb{R}^n$. Plug into $\dot{x} = Ax$

$$\left.\begin{aligned} \dot{x} &= \lambda e^{\lambda t} v \\ Ax &= e^{\lambda t} A v \end{aligned}\right\} \Rightarrow \lambda e^{\lambda t} v = e^{\lambda t} A v \quad \text{then}$$

$$\boxed{\lambda v = A v}$$

Obs: $x = e^{\lambda t} v$ is a solution of $\dot{x} = Ax$ if and only if $\lambda v = Av$ (λ eigenvalue & v eigenvector)

Goal: Find n linearly independent solutions of

$\dot{x} = Ax$

Obs: Linearly independent eigenvectors give us linearly independent solutions. Sometimes, we do not have n linearly independent eigenvectors.

Obs: a & λ numbers $a, \lambda \in \mathbb{C}$

$\dot{x} = ax$, $x \in \mathbb{R}$ $x = x(t)$ $v \in \mathbb{C}$

$$x = e^{at} v = e^{(a-\lambda)t} e^{\lambda t} v = e^{\lambda t} e^{(a-\lambda)t} v =$$

$$= e^{\lambda t} \left(\sum_{k=0}^{\infty} \frac{(a-\lambda)^k t^k}{k!} \right) v = e^{\lambda t} \left[\sum_{k=0}^{\infty} \frac{t^k (a-\lambda)^k}{k!} v \right]$$

Replace a by A

$$x = e^{\lambda t} \left[\sum_{k=0}^{\infty} \frac{t^k}{k!} (A - \lambda I)^k v \right]$$

Th: Let $P(\lambda) = \det(A - \lambda I) = (-1)^n (\lambda - \lambda_1)^{n_1} \cdots \cdots (\lambda - \lambda_r)^{n_r}$. $\lambda_i \neq \lambda_j$ $i \neq j$. There are ~~at most~~ n_i linearly independent solutions of

$$(A - \lambda_i I)^{n_i} v = 0$$

Example $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $P(\lambda) = -(\lambda-1)^2(\lambda-2)$

<u>Example</u> $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $p(\lambda) = -(\lambda$

$\lambda_1 = 1 \quad n_1 = 2$

$\lambda_2 = 2 \quad n_2 = 1$

$\boxed{\lambda_1 = 1}$ $(A - \lambda_1 I)v = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} v = 0 \quad v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$(A - \lambda_1 I)^2 v = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2 v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} v = 0$$

$$v = t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are 2 linearly independent solutions

to $(A - \lambda_1 I)^2 v = 0$

Steps to find n linearly independent solutions to $\dot{x} = Ax \qquad A \in \mathbb{R}^{n \times n} \quad x = x(t) \in \mathbb{R}^n$

<u>Step 1</u>: Find the eigenvalues $\lambda_1, \ldots, \lambda_r$ and their algebraic multiplicities $n_1, \ldots, n_r$.

<u>Step 2</u>: For each eigenvalue λ_i find n_i linearly independent solutions of $(A - \lambda_i I)^{n_i} v = 0$. Call these solutions $v_1^{(i)}, \ldots, v_{n_i}^{(i)}$

Step 3: For each eigenvalue λ_i, set

$$\boxed{x_\ell^{(i)} = e^{\lambda_i t} \sum_{k=0}^{n_i-1} \frac{t^k}{k!} (A-\lambda_i I)^k v_\ell^{(i)}}$$

$1 \le \ell \le n_i$

These $x_\ell^{(i)}$ $\begin{pmatrix} 1 \le i \le r \\ 1 \le \ell \le n_i \end{pmatrix}$ form a set of n linearly independent solutions.

Step 4: If some eigenvalues are complex, keep only one per pair of complex conjugates. Once you have all the solutions, take real and imaginary parts of all the complex valued solutions

Step 5: The general solution is any linear combination of the n solutions obtained.

Step 6: Impose any given initial conditions.

Examples: 1) $\dot{x}_1 = -x_1 + x_2$ $\quad x_1(0) = 1$

$\dot{x}_2 = 3x_2$ $\quad x_2(0) = 2$

$A = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix}$ $\quad \lambda_1 = -1 \quad n_1 = 1$

$\lambda_2 = 3 \quad n_2 = 1$

$\lambda_1 = -1 \quad (A - \lambda_1 I)^1 = \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\lambda_1 = -1 \quad (A - \lambda_1 I)^1 = \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 3 \quad (A - \lambda_2 I)^1 = \begin{bmatrix} -4 & 1 \\ 0 & 0 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Convention or notation $B^0 = I$ for any $B \in \mathbb{R}^{n \times n}$

General solution $X = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$$X(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$c_1 = c_2 = \frac{1}{2} \qquad X = \frac{1}{2} e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2} e^{3t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$x_1 = \frac{1}{2} e^{-t} + \frac{1}{2} e^{3t}$$

$$x_2 = 2 e^{3t}$$

<u>Example</u>: $\dot{x}_1 = x_1 - 4x_2$

$\dot{x}_2 = x_1 + x_2$

$\dot{x}_3 = -2x_3$

$$A = \begin{bmatrix} 1 & -4 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P(\lambda) = \det \begin{bmatrix} 1-\lambda & -4 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda)(-2-\lambda) + 4(-2-\lambda)$$

$$P(\lambda) = (-1)(\lambda+2)\left[(\lambda-1)^2 + 4\right] = 0$$

$p(\lambda) = (-1)(\lambda+2)[(\lambda-1)^2+4] = 0$

$\lambda_1 = -2 \qquad (\lambda-1)^2+4=0 \Rightarrow (\lambda-1)^2=-4 \qquad \lambda-1=\pm 2i$

$\lambda = 1 \pm 2i$

$\lambda_2 = 1+2i \qquad n_1 = 1 \quad n_2 = 1 \quad n_3 = 1$

$\lambda_3 = 1-2i$

$$\boxed{\lambda_1 = -2} \qquad A - \lambda_1 I = \begin{bmatrix} 3 & -4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\lambda_2 = 1+2i} \qquad A - \lambda_2 I = \begin{bmatrix} -2i & -4 & 0 \\ 1 & -2i & 0 \\ 0 & 0 & -3-2i \end{bmatrix} \qquad v = \begin{bmatrix} 2i \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_1 \longrightarrow e^{-2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 \longrightarrow e^{(1+2i)t} \begin{bmatrix} 2i \\ 1 \\ 0 \end{bmatrix} = e^t \begin{bmatrix} 2i\, e^{2it} \\ e^{2it} \\ 0 \end{bmatrix} = e^t \begin{bmatrix} -2\sin 2t \\ \cos 2t \\ 0 \end{bmatrix} + i e^t \begin{bmatrix} 2\cos 2t \\ \sin 2t \\ 0 \end{bmatrix}$$

$$\longrightarrow e^t \begin{bmatrix} -2\sin 2t \\ \cos 2t \\ 0 \end{bmatrix} \text{ and } e^t \begin{bmatrix} 2\cos 2t \\ \sin 2t \\ 0 \end{bmatrix}$$

General solution: $x = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -2\sin 2t \\ \cos 2t \\ 0 \end{bmatrix} e^t + c_3 \begin{bmatrix} 2\cos 2t \\ \sin 2t \\ 0 \end{bmatrix} e^t$

Example: $\dot{x}_1 = 2x_1 + x_2$

$\dot{x}_2 = 2x_2 + x_3$

$\dot{x}_3 = x_3$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P(\lambda) = -(\lambda-2)^2(\lambda-1)$$

$\lambda_1 = 1$	$\lambda_2 = 2$
$n_1 = 1$	$n_2 = 2$

$$\lambda_1 = 1 \quad A - \lambda_1 I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \boxed{x_1 = e^{t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$$

$$\lambda_2 = 2 \quad (A - \lambda_2 I)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$x_2 = e^{\lambda_2 t}\left(\sum_{k=0}^{n_2-1} \frac{t^k}{k!}(A - \lambda_2 I)^k v_1\right)$$

$$x_2 = e^{\lambda_2 t}\left(v_1 + t(A - \lambda_2 I)v_1\right) =$$

$$x_2 = e^{2t}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x_3 = e^{\lambda_2 t}\left(\sum_{k=0}^{n_2-1} \frac{t^k}{k!}(A - \lambda_2 I)^k v_2\right)$$

$$x_3 = e^{2t}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = e^{2t}\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$$

General solution $x = c_1 e^{t}\begin{bmatrix}1\\-1\\1\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}1\\0\\0\end{bmatrix} + c_3 e^{2t}\begin{bmatrix}t\\1\\0\end{bmatrix}$

<u>Non-linear autonomous systems</u> (1st order)

$$(1)\quad \begin{cases} \dot{x}_1 = f_1(x_1, \dots, x_n) \\ \dot{x}_2 = f_2(x_1, \dots, x_n) \\ \vdots \qquad\quad \vdots \\ \dot{x}_n = f_n(x_1, \dots, x_n) \end{cases}$$

<u>Example</u> 1) $\dot{x}_1 = x_1(1-x_2)$
$\dot{x}_2 = -3x_1^2 + x_2$

2) $\dot{x}_1 = x_1^2 + t$
$\dot{x}_2 = x_1(1-x_2)$ is not autonomous

<u>Obs</u>: We can write the system (1) in vector form $\dot{x} = F(x)$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad F(x) = \begin{bmatrix} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) \end{bmatrix}$$

<u>Obs</u>: $x_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ \vdots \\ x_{0n} \end{bmatrix}$ independent of t, is a solution of $\dot{x} = F(x)$ if and only

$$\begin{bmatrix} \vdots \\ x_{0n} \end{bmatrix} \text{ of } \dot{x} = F(x) \text{ if and only}$$

if $0 = F(x_0)$, i.e.

$$f_1(x_{01}, \dots, x_{0n}) = 0$$

$$\vdots$$

$$f_n(x_{01}, \dots, x_{0n}) = 0$$

Def: these constant solutions x_0 are called fixed points or equilibrium solutions.

Example: Find the fixed points of

$$\dot{x}_1 = x_2(1-x_1)$$

$$\dot{x}_2 = x_2 - x_2^2$$

$$x_2(1-x_1) = 0$$

$$x_2 - x_2^2 = 0 \longrightarrow x_2 = 0 \text{ or } x_2 = 1$$

If $x_2 = 0$ then $x_1 = c$ $\quad c \in \mathbb{R}$

If $x_2 = 1$ then $x_1 = 1$

Fixed points $\begin{bmatrix} c \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for any $c \in \mathbb{R}$.