$f$ analytic around $z_0$ but not defined at $z_0$. We say that $z_0$ is a singularity of $f$. Let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{Laurent series}$$

| Type of singularity | Laurent series | Example |
|---|---|---|
| Removable | $a_0 + a_1(z-z_0) + \cdots$ | $\frac{\sin z}{z}$ |
| Pole of order $n$, $n>0$ | $a_{-n}(z-z_0)^{-n} + \cdots$ | $\frac{1}{(z-z_0)^n}$ |
| Simple pole | $a_{-1}(z-z_0)^{-1} + \cdots$ | $\frac{1}{(z-z_0)}$ |
| Essential singularity | $\cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + \cdots$ | $e^{\frac{1}{z}}$ |

Def: $f$ analytic at $z_0$. $f$ has a zero of order $n$ at $z_0$ if $f(z) = a_n(z-z_0)^n + \cdots$ $\quad a_n \neq 0$

Theorem: $f$ & $g$ are analytic at $z_0$. $f$ has a zero of order $n$ at $z_0$, $n>0$. $g(z_0)\neq 0$. Let $F(z) = \frac{g(z)}{f(z)}$ has a pole of order $n$ at $z_0$

**Example** $F(z) = \dfrac{2z+5}{(z-1)(z+5)(z-2)^4}$

$g = \dfrac{2z+5}{(z+5)(z-2)^4}$ $f = (z-1)$

2 is a pole of order 4

1 is a simple pole

$-5$ is a simple pole

**Reminder**: $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ then $\mathrm{Res}(f, z_0) = a_{-1}$

**theorem**: If $f$ has a simple pole at $z_0$, then
$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z-z_0) f(z_0)$

**proof**: $f(z) = \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$

$$\lim_{z \to z_0} (z-z_0) f(z) = \lim_{z \to z_0} a_{-1} + a_0(z-z_0) + \cdots = a_{-1}$$

**Example** $f(z) = \dfrac{a_{-3}}{(z-z_0)^3} + \dfrac{a_{-2}}{(z-z_0)^2} + \dfrac{a_{-1}}{(z-z_0)} + \cdots$

$$g(z) = (z-z_0)^3 f(z) = a_{-3} + a_{-2}(z-z_0) + a_{-1}(z-z_0)^2 + \cdots$$

$$g(z) = g(z_0) + g'(z_0)(z-z_0) + \frac{g''(z_0)}{2!}(z-z_0)^2 + \cdots$$

$$\lim_{z \to z_0} \frac{1}{2!} \frac{d^2}{dz^2}\left((z-z_0)^3 f(z)\right) = a_{-1}$$

<u>Theorem</u>: $f$ has a pole of order $n$ at $z_0$. then

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \frac{1}{(n-1)!}$$

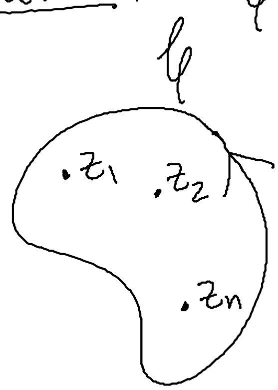
<u>Example</u>: $f(z) = \dfrac{1}{(z-1)^2(z-3)}$

$$\mathrm{Res}(f, 3) = \lim_{z \to 3} (z-3) f(z) = \lim_{z \to 3} \frac{1}{(z-1)^2} = \frac{1}{4}$$

$$\mathrm{Res}(f, 1) = \lim_{z \to 1} \frac{1}{1!} \frac{d}{dz} (z-1)^2 f(z) = \lim_{z \to 1} \frac{d}{dz} \frac{1}{z-3} =$$

$$= \left.\frac{(-1)}{(z-3)^2}\right|_{z=1} = \frac{-1}{4}$$

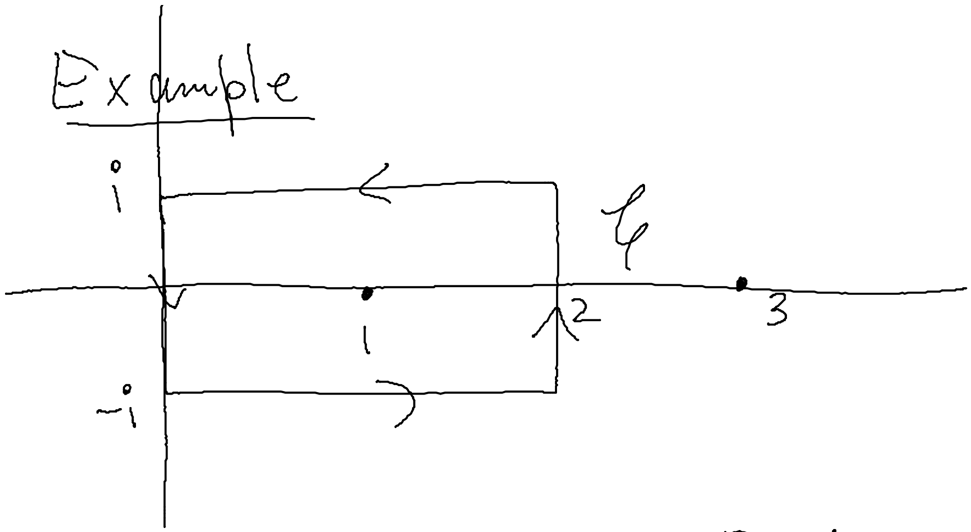
<u>Theorem</u>: $\mathcal{C}$ simple closed curve. $z_1, z_2, \ldots, z_n$ inside $\mathcal{C}$. $f$ analytic at $\mathcal{C}$ and inside except at $z_1, z_2, \ldots, z_n$. then




$$\oint_{\mathcal{C}} f(z)\, dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k)$$

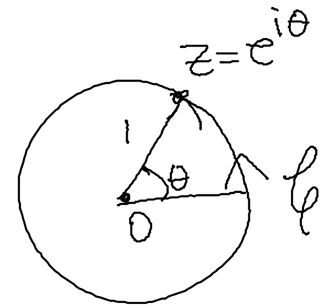
Example

## Example




$$\oint_{\mathcal{C}} \frac{1}{(z-1)^2(z-3)}\,dz = 2\pi i \,\mathrm{Res}(f,1) = 2\pi i\left(\frac{-1}{4}\right) = -\frac{\pi i}{2}$$

Integrals of the form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta$




$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$$

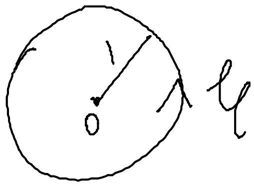
$$dz = ie^{i\theta}\,d\theta$$

$$d\theta = -i\,z^{-1}\,dz$$

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta =$$

$$= \oint_{\mathcal{C}} F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{(-i)}{z}\,dz$$

Example: $\displaystyle\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2} = \oint_{|z|=1} \frac{(-i)\,z^{-1}}{\left(2+\frac{z+z^{-1}}{2}\right)^2}\,dz =$

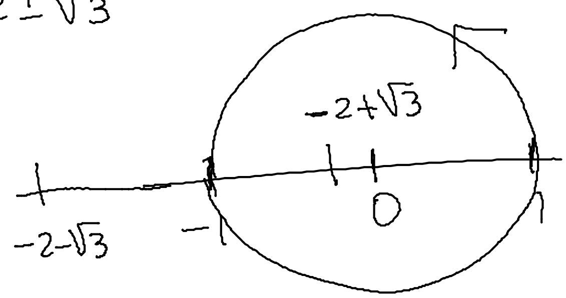



$$= \oint_{|z|=1} f(z)\,dz$$

$$f(z) = \frac{1}{iz\left(2+\frac{z+\frac{1}{z}}{2}\right)^2} = \frac{4z^2}{iz\left(4z+z^2+1\right)^2} = -i4\frac{z}{(z^2+4z+1)^2} =$$

$$\frac{-4\pm\sqrt{16-4}}{2} = \frac{-4\pm\sqrt{12}}{2} = -2\pm\sqrt{3}$$

$$f(z) = \frac{-i4\ z}{\left(z-(-2+\sqrt{3})\right)^2\left(z-(-2-\sqrt{3})\right)^2}$$




$$\operatorname{Res}(f,-2+\sqrt{3}) = \lim_{z\to-2+\sqrt{3}} \frac{1}{1!}\frac{d}{dz} f(z)\left(z-(-2+\sqrt{3})\right)^2 = \frac{d}{dz}\left(\frac{-4iz}{\left(z-(-2-\sqrt{3})\right)^2}\right)\Bigg|_{z=-2+\sqrt{3}}$$

$$= -4i\left\{\frac{\left(z-(-2-\sqrt{3})\right)^2 - z\,2\left(z-(-2-\sqrt{3})\right)}{\left(z-(-2-\sqrt{3})\right)^4}\right\}\Bigg|_{z=-2+\sqrt{3}}$$

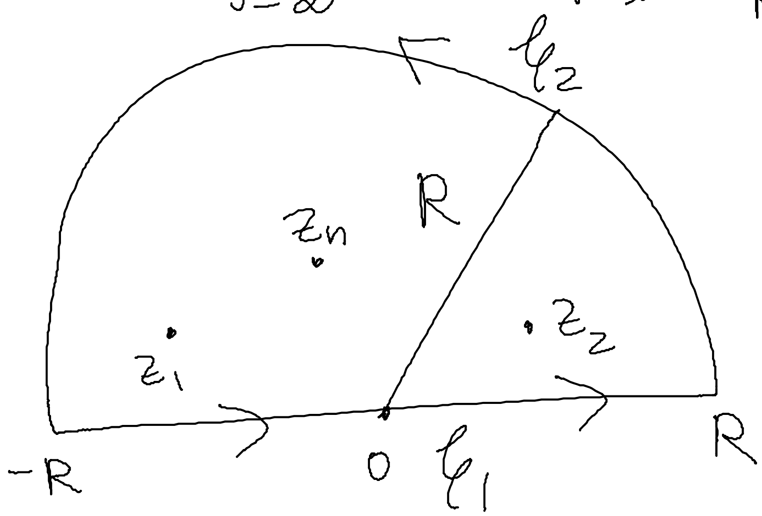
$$= -4i\left\{\frac{(2\sqrt{3})^2-2(-2+\sqrt{3})(2\sqrt{3})}{(2\sqrt{3})^4}\right\} = \frac{-4i(2)^2\sqrt{3}\left(\sqrt{3}+2-\sqrt{3}\right)}{16(9)} =$$

$$= -\frac{i\sqrt{3}}{9}\,2$$

$$\boxed{\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2} = 2\pi i \,\mathrm{Res}\left(f, -2+\sqrt{3}\right) = 2\pi i\left(-\frac{i\sqrt{3}\,2}{9}\right) = \frac{4\pi\sqrt{3}}{9}}$$

<u>Integrals of the form</u> $\int_{-\infty}^{\infty} f(x)\,dx$

<u>Reminder</u>: $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{0} f(x)\,dx + \lim_{R\to\infty} \int_{0}^{R} f(x)\,dx$




$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$

$$\oint_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{C}_1} f(z)\,dz + \int_{\mathcal{C}_2} f(z)\,dz$$

If $\lim_{R\to\infty} \int_{\mathcal{C}_2} f(z)\,dz = 0$, then

$$\lim_{R\to\infty} \oint_{\mathcal{C}} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$$

$$\lim_{R\to\infty} \oint_{\mathcal{C}} f(z)\,dz = 2\pi i \sum_{\substack{z_k \text{ singularity of } f \\ \text{and } \operatorname{Im} z_k > 0}} \mathrm{Res}(f, z_k)$$

and $\text{Im } z_k > 0$

Assume $f$ has no singularities in the real line.

<u>Obs</u>: $f(z) = \dfrac{P(z)}{Q(z)}$. $P$ & $Q$ polynomials. If degree of $Q \geq$ degree of $P + 2$ then

$$\lim_{R\to\infty} \int_{\substack{|z|=R \\ \text{Im}(z)>0}} f(z)\,dz = 0$$

<u>proof</u>: $P(z) = a_n z^n + \cdots + a_0$

$Q(z) = b_k z^k + \cdots + b_0$

$$\left| \int_{\substack{|z|=R \\ \text{Im}(z)>0}} \frac{P(z)}{Q(z)}\,dz \right| \cong \left| \int_{\substack{|z|=R \\ \text{Im}(z)>0}} \frac{a_n z^n}{b_k z^k}\,dz \right| = \left|\frac{a_n}{b_k}\right| \left| \int_{\substack{|z|=R \\ \text{Im}(z)>0}} \frac{1}{z^{k-n}}\,dz \right|$$

$$\leq \left|\frac{a_n}{b_k}\right| \int_{\substack{|z|=R \\ \text{Im}(z)>0}} \underbrace{\left|\frac{1}{z^{k-n}}\right|}_{\frac{1}{R^{k-n}}} |dz| = \left|\frac{a_n}{b_k}\right| \frac{1}{R^{k-n}} \pi R$$

$$\frac{1}{R^{k-n-1}} \to 0$$

$k > n+1$

Example: $$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+9)} = 2\pi i \sum_{\text{Im}(z_k)>0} \text{Res}(f, z_k)$$

$P(z) = 1$

$Q(z) = (z^2+1)(z^2+9)$

$\deg Q = 4 > \underbrace{\deg P}_{=0} + 2 = 2$

Singularities: roots of $Q$: $-i, i, -3i, 3i$

$z_1 = i \qquad z_2 = 3i$

$$\mathrm{Res}(f, i) = \lim_{z \to i} (z-i) f(z) = \left. \frac{\cancel{(z-i)}}{\cancel{(z-i)}(z+i)(z^2+9)} \right|_{z=i} = \frac{1}{16i}$$

$$\mathrm{Res}(f, 3i) = \left. \frac{\cancel{(z-3i)}}{(z^2+1)(z+3i)\cancel{(z-3i)}} \right|_{z=3i} = -\frac{1}{48i}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+9)}} = 2\pi i \left\{ \frac{1}{16i} - \frac{1}{48i} \right\} = \pi \left( \frac{1}{8} - \frac{1}{24} \right) = \boxed{\frac{\pi}{12}}$$