MATH 6701, SUMMER 2016.

WILFRID GANGBO

1. Solutions to test

Below, P_2 is a the set of all polynomials of degree less than or equal to 2. On P_2 we define the inner product

$$(p, q) = \int_{-1}^{1} p(x)q(x)dx.$$

Exercise 1.1. *Let $\mathcal{P}$ be a plane that contains $P_1(3,4,-5)$ and $P_2(-2,8,-5)$.*

(i) The only plane that is parallel to the coordinate yz–plane is $z=-5$.

(ii) If $\mathcal{P}$ is parallel to the x–axis then it is parallel to both $a=\mathbf{i}$ and $b = P_1 - P_2 = (5,-4,0) = 5\mathbf{i} - 4\mathbf{j}$. Thus $\mathcal{P}$ is perpendicular to $a \times b = -4\mathbf{k}$. Thus, the equation of $\mathcal{P}$ is $z = d$ and so, since P_1 belongs to the plane, $d=-5$.

Exercise 1.2. *(i) The distance from a point $P(0,y,z)$ in the yz–plane and $(7,-3,-4)$ is $||\vec{MP}|| = \sqrt{49+(y+3)^2+(z+4)^2}$. Its smallest value is attained when $y=-3$ and $z=-4$. Therefore, 7 is the distance from the point $M(7,-3,-4)$ to the yz–plane.*

(ii) The distance between a point $P(x,0,0)$ on the x–axis and $(7,-3,-4)$ is $||\vec{MP}|| = \sqrt{(x-7)^2+9+16}$. Its smallest value is attained when $x=7$. Therefore, 5 is the distance from the point $M(7,-3,-4)$ to the x–axis. .

Exercise 1.3. *Find* $\text{proj}_{a+b}a$ *Assume $a = 4\mathbf{i}+3\mathbf{j}$ and $b=-\mathbf{i}+\mathbf{j}$ and set $c = a+b = 3\mathbf{i}+4\mathbf{j}$. We have $||c||^2 = 9+14 = 25$ and $c \cdot a = 12+12 = 24$. Thus,*

$$\text{proj}_{a+b}a = \frac{c \cdot a}{||c||^2}c = \frac{24}{25}c = \frac{72}{25}\mathbf{i} + \frac{96}{25}\mathbf{j}.$$

Exercise 1.4 (30 points)**.** *The area of the triangle Δ determined by $P_1(1,0,3)$, $P_2(0,0,6)$, $P_3(2,4,5)$ is* $\text{area}(\Delta) = 1/2||\vec{P_1P_2} \times \vec{P_1P_3}||$. *But*

$$\vec{P_1P_2} \times \vec{P_1P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 3 \\ 1 & 4 & 2 \end{vmatrix} = -12\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$$

Thus $\text{area}(\Delta) = 1/2\sqrt{144+25+16} = \sqrt{185}/2$.

Find the area of the triangle determined by $P_1(1,0,3)$, $P_2(0,0,6)$, $P_3(2,4,5)$.

Exercise 1.5. *Let* $P_1(1,1,-2)$, $P_2(4,0,-3)$, $P_3(1,-5,10)$ *and* $P_4(-7,2,4)$. *We have*

$$\det\big(\vec{P_1P_2},\vec{P_1P_3},\vec{P_1P_4}\big)=\begin{vmatrix}3&-1&-1\\0&-6&12\\-8&1&6\end{vmatrix}=3\begin{vmatrix}-6&12\\1&6\end{vmatrix}-8\begin{vmatrix}-1&-1\\-6&12\end{vmatrix}=-3\times48+8\times18=0.$$

The four points don't generate a positive volume, therefore they lie in the same plane.

Exercise 1.6. *Consider the lines* $(x,y,z)=(1+t,2-t,3t)$ *and* $(x,y,z)=(2-s,1+s,6s)$. *They intersect if and only if we can find* s *and* t *such that* $(1+t,2-t,3t)=(2-s,1+s,6s)$. *The latter equation give* $3t=6s$ *and so* $t=2s$. *We use that in the first two equations to get* $t=2/3$ *and conclude that the two lines intersect at* $(5/3,4/3,2)$.

Exercise 1.7. *The plane* $\mathcal{P}$ *that contains the lines* $(x,y,z)=(1+3t,1-t,2+t)$, *and* $(x,y,z)=(4+4s,2s,3+s)$ *is parallel to* $a=(3,-1,1)$ *and* $b=(4,2,1)$ *and so, it is perpendicular to*

$$n=a\times b=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&-1&1\\4&2&1\end{vmatrix}=-3\mathbf{i}+\mathbf{j}+10\mathbf{k}.$$

Thus, the equation of the plane is $-3x+y+10z=d$. *Since* $(1,1,2)$ *belongs to* $\mathcal{P}$, *we have* $-3+1+20=d$ *and so,* $d=18$.

Exercise 1.8. *Let* $B=\{p_1(x)=x^2-x,p_2(x)=x^2+1,p_3(x)=1-x^2\}$ *be a basis of* $\mathcal{P}_2$. *In the calculation below, we will use the following facts:*

$$\int_{-1}^1 x^{2n+1}dx=0,\quad \int_{-1}^1 x^{2n}dx=2\int_0^1 x^{2n}dx=\frac{2}{2n+1}.$$

We have

$$(p_1,p_2)=\int_{-1}^1(x^2-x)(x^2+1)dx=\int_{-1}^1(x^4+x^2)dx=\frac{2}{5}+\frac{2}{3}=\frac{16}{15}$$

and

$$||p_1||^2=\int_{-1}^1(x^2-x)^2dx=\int_{-1}^1(x^4+x^2)dx=\frac{2}{5}+\frac{2}{3}=\frac{16}{15}.$$

We set

$$q_2=p_2-\frac{(p_1,p_2)}{||p_1||^2}p_1=p_2-p_1=x+1.$$

We have

$$(q_1,p_3)=\int_{-1}^1(x^2-x)(1-x^2)dx=\int_{-1}^1(x^2-x^4)dx=\frac{2}{3}-\frac{2}{5}=\frac{4}{15},$$

$$(q_2,p_3)=\int_{-1}^1(x+1)(1-x^2)dx=\int_{-1}^1(1-x^2)dx=2-\frac{2}{3}=\frac{4}{3}$$

and

$$||q_2||^2 = \int_{-1}^{1}(x+1)^2dx = \int_{-1}^{1}(x^2+1)dx = 2 + \frac{2}{3} = \frac{8}{3}.$$

We set

$$q_3 = p_3 - \frac{(q_1, p_3)}{||q_1||^2}q_1 - \frac{(q_2, p_3)}{||q_2||^2}q_2 = p_3 - \frac{1}{4}q_1 - \frac{1}{2}q_2.$$

By the Gram–Schmidt orthogonalization process, $B' = \{q_1, q_2, q_3\}$ *is an orthogonal basis of* $\mathcal{P}_2$

Georgia Institute of Technology, Atlanta GA, USA

E-mail address: gangbo@math.gatech.edu