Note Title 7/6/2016

Inhomogeneous linear constant coefficients odes

(2) $$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = x^{l} e^{\beta x}$$

$a_0, \dots, a_{n-1} \in \mathbb{R}$ $\quad l \geq 0$ integer $\quad \beta \in \mathbb{C}$

let $k = \begin{cases} \text{multiplicity of } \beta \text{ as root } P(\lambda) \\ 0 \text{ if } P(\beta) \neq 0 \end{cases}$

$$P(\lambda) = \lambda^{n} + a_{n-1}\lambda^{n-1} + \dots + a_1 \lambda + a_0$$

Set $\boxed{y = x^{k} (c_0 + c_1 x + \dots + c_l x^{l}) \, e^{\beta x}}$

Plug into eq(2) and find $c_0, \dots, c_\ell$

Example: 1) $y' + y = e^{3x}$

$\beta = 3$	$P(\lambda) = \lambda + 1$	$y = c_0 e^{3x}$	$c_0 = \frac{1}{4}$
$\ell = 0$	$P(\beta) = 4 \neq 0$	Plug int eq	$y = \frac{e^{3x}}{4}$
$k = 0$		$c_0 3 e^{3x} + c_0 e^{3x} = e^{3x}$	

2) $y'' + y = x e^{ix}$

$\beta = i$	$P(\lambda) = \lambda^2 + 1$	$y = x(c_0 + c_1 x) e^{ix}$
$\ell = 1$	$P(i) = 0$	$y' = (c_0 + 2c_1 x + i c_0 x + i c_1 x^2) e^{ix}$
$k = 1$	$P(\lambda) = (\lambda - i)^1 (\lambda + i)^1$	$y' = (c_0 + (2c_1 + i c_0) x + i c_1 x^2) e^{ix}$

$$y'' = (2C_1 + iC_0 + 2iC_1 X)\, e^{iX} + (iC_0 + (2iC_1 - C_0) X - C_1 X^2)\, e^{iX}$$

$$y'' = \left((2C_1 + 2iC_0) + (4iC_1 - C_0)\, X - C_1 X^2\right) e^{iX}$$

Plug into eq & devide by e^{iX}

$$(2C_1 + 2iC_0) + (4iC_1 - C_0) X - C_1 X^2 + C_0 X + C_1 X^2 = X$$

$$\underbrace{(2C_1 + 2iC_0)}_{=0} + \underbrace{4iC_1}_{=1} X = X$$

$$C_1 = \frac{1}{4i} = \frac{-i}{4}$$

$$C_0 = \frac{-C_1}{i} = \frac{1}{4}$$

$$y = X\left(\frac{1}{4} - \frac{i}{4} X\right) e^{iX}$$

<u>Obs</u>: $f(x) = f_1(x) + i f_2(x)$. Let $y = y_1 + i y_2$. Then y is a solution of

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f$$

if and only if y_1 solves

$$y_1^{(n)} + a_{n-1} y_1^{(n-1)} + \cdots + a_1 y_1' + a_0 y_1 = f_1$$

and y_2 solves

$$y_2^{(n)} + a_{n-1} y_2^{(n-1)} + \cdots + a_1 y_2' + a_0 y_2 = f_2$$

<u>Ex</u>: $y'' + y = x\cos x$ Note $x\cos x = \mathrm{Re}(xe^{ix})$

1st Solve $y'' + y = xe^{ix}$. One solution is $y = \frac{x}{4}(1 - ix)e^{ix}$

Then, a solution of $y''+y=x\cos x$ is $\underset{y_1}{\underbrace{Re(y)}} = Re\left(\frac{x(1-ix)}{4}e^{ix}\right)$

$y_1 = \frac{x}{4}\cos x + \frac{x^2}{4}\sin x$

<u>Obs</u>: To find a solution of

$y^{(n)} + \cdots + a_0 y = x^l e^{ax}\cos(bx)$

Use the fact that $x^l e^{ax}\cos(bx) = Re\left[x^l e^{(a+bi)x}\right]$

<u>Ex</u>: $y' + 2y = \sin(3x) = Im\left(e^{3ix}\right)$

Solve $z' + 2z = e^{3ix}$

$\beta = 3i$

$l = 0$

$k = 0$

$P(\lambda) = \lambda + 2$

$P(\beta) = 3i + 2 \neq 0$

$z = A e^{3ix}$

$3iA e^{3ix} + 2A e^{3ix} = e^{3ix}$

$A = \frac{1}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{(2-3i)}{13}$

$z = \left(\frac{2}{13} - \frac{3}{13} i\right) e^{3ix}$

$y = \mathrm{Im}(z)$

$$\boxed{y = \frac{2}{13} \sin 3x - \frac{3}{13} \cos 3x}$$

Obs: If $y_1^{(n)} + \dots + a_0 y_1 = f_1$

and $y_2^{(n)} + \dots + a_0 y_2 = f_2$

then if $y = c_1 y_1 + c_2 y_2$, then y solves

$$y^{(n)} + \cdots + a_0 y = c_1 f_1 + c_2 f_2$$

We can find a solution $y^{(n)} + \cdots + a_0 y = f$, where f is a linear combination of products of powers of x, exponentials, sines or cosines

<u>Ex</u> We can find a solution of

$$y'' + y' + 2y = 2x e^{-3x} - 7x^2 \sin(5x) e^{2x} + 2 \cos 8x$$

<u>theorem</u> Let y_p be a solution of

(1) $y^{(n)} + \cdots + a_0 y = f$

then y is a solution of eq (1) if $y = y_p + y_h$, where y_h is a solution of

$$y_h^{(n)} + \cdots + a_0 y_h = 0$$

So now we can find all the solutions to eq (1)

Example $y'' - 3y' + 2y = x e^x \sin x$

$z'' - 3z' + 2z = x e^{(1+i)x}$ because $x e^x \sin x = \operatorname{Im}\left[x e^{(1+i)x}\right]$

$\beta = 1+i$ | $P(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$

$l = 1$ | $z = (a+bx)\, e^{(1+i)x}$

$k = 0 \quad | \quad z' = \{[a(1+i)+b] + b(1+i)x\} e^{(1+i)x}$

$$z'' = \left(2ia + 2b(1+i) + 2ibx\right) e^{(1+i)x}$$

Plug into eq & devide by $e^{(1+i)x}$

$$2ia + 2b(1+i) + 2ibx - 3a(1+i) - 3b - 3b(1+i)x + 2a + 2bx = x$$

$$-(1+i)a + (-1+2i)b - (1+i)xb = x$$

$-(1+i)b = 1 \qquad -(1+i)a + (-1+2i)b = 0$

$$b = \frac{-1}{1+i} \cdot \frac{1-i}{1-i} = \frac{-1+i}{2} = -\frac{1}{2} + \frac{i}{2}$$

$$-(1+i)\,a + \frac{(-1+2i)}{2}\,\frac{(-1+i)}{2i} = 0 \qquad a = \frac{(-1+2i)}{2}\,\frac{(-1+i)}{1+i}\cdot\frac{1-i}{1-i}$$

$$a = \frac{(-1+2i)\overbrace{(-1+i)\,(1-i)}^{=\,2i}}{4} = \frac{1}{2}(2+i) = 1+\frac{i}{2}$$

$$z = \left[\left(1+\frac{i}{2}\right)+\left(-\frac{1}{2}+\frac{1}{2}\right)x\right] e^{(1+i)x}$$

$$y_p = \operatorname{Im}(z) = \left\{\left(1-\frac{1}{2}x\right)\sin x + \frac{(1+x)}{2}\cos x\right\} e^{x}$$

All the solutions are of the form

$$\boxed{y = \left(\left(1 - \tfrac{1}{2}x\right)\sin x + \frac{(1+x)}{2}\cos x \right) e^{x} + c_1 e^{x} + c_2 e^{2x} \qquad c_1, c_2 \in \mathbb{R}}$$

Homogeneous systems of 1^{st} order linear constant coefficients

$$
\begin{aligned}
\dot{x}_1 &= a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \\
\dot{x}_2 &= a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \\
&\vdots \\
\dot{x}_n &= a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n
\end{aligned}
$$

$a_{ij} \in \mathbb{R}$

a_{ij} known

Goal

find $x_i(t)$

Matrix notation

$$x = x(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\boxed{\dot{x} = Ax}$$

<u>Obs</u>: Let $x = e^{\lambda t} v$, $\lambda \in \mathbb{C}$, $v \in \mathbb{C}^n$

$\dot{x} = \lambda e^{\lambda t} v$ Since $\dot{x} = Ax$, then if $x = e^{\lambda t} v$,

$Ax = e^{\lambda t} Av$

$\lambda e^{\lambda t} v = e^{\lambda t} A v$, thus $\boxed{\lambda v = Av}$

<u>Obs</u>: $x = e^{\lambda t} v$ is a solution of $\dot{x} = Ax$ if and only if $\lambda v = Av$, i.e. v is an eigenvector with eigenvalue λ.

<u>Recall</u>: $P(\lambda) = \det\left[A - \lambda I\right]$

λ is an eigenvalue of A if and only if $P(\lambda) = 0$

v is an eigenvector if and only if $v \neq 0$ and $(A - \lambda I)v = 0$

<u>Goal</u>: Find n linearly independent solutions of $\dot{x} = Ax$

<u>Obs</u>: 1) Linearly independent eigenvectors give us linearly independent solutions.

2) Sometimes, we do not have n linearly independent eigenvectors

<u>Obs</u>: a & $\lambda \in \mathbb{C}$

$\dot{x} = ax$, then $x = v e^{at}$ $v \in \mathbb{C}$

$$x = e^{at} v = e^{\lambda t}\left[e^{(a-\lambda)t}\right] v = e^{\lambda t}\left[\sum_{k=0}^{\infty} \frac{(a-\lambda)^k t^k}{k!}\right] v$$

$$x = e^{\lambda t}\left[\sum_{k=0}^{\infty} \frac{t^k}{k!}(a-\lambda)^k v\right]$$

Replace a by A

$$\dot{x} = Ax \qquad x = e^{\lambda t}\left[\sum_{k=0}^{\infty} \frac{t^k}{k!}(A-\lambda I)^k v\right]$$ need to make sense of this

<u>Theorem</u>: $P(\lambda) = \det(A-\lambda I) = (-1)^n(\lambda-\lambda_1)^{n_1}\cdots\cdots(\lambda-\lambda_r)^{n_r}$

$\lambda_i \neq \lambda_j$ if $i \neq j$ $\quad n_1 + n_2 + \cdots + n_r = n$

There are n_i linearly independent solutions to

$$(A-\lambda_i I)^{n_i} v = 0$$

Example $\quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad P(\lambda) = (-1)(\lambda-1)^2(\lambda-2)$

$\lambda_1 = 1 \qquad n_1 = 2$

$\lambda_2 = 2 \qquad n_2 = 1$

$\lambda_1 = 1$: $\quad A - \lambda_1 I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underbrace{(A-\lambda_1 I)}_{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_2 \\ 0 \\ v_3 \end{bmatrix}} v = 0 \qquad v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$(A-\lambda_1 I)^{n_1} v = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2 v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} v \qquad v = t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are two linearly independent solutions of

$$(A-\lambda_1 I)^2 v = 0 \qquad 2 = n_1$$