Th: Assume: 1) f has a finite number of singularities in $\operatorname{Im}(z) \geq 0$

2) Let $M_R = \max\limits_{\substack{|z|=R \\ \operatorname{Im}(z)\geq 0}} |f(z)|$. Assume

$$\lim_{R\to\infty} M_R = 0.$$

Then $\displaystyle\lim_{R\to\infty} \int_{\substack{|z|=R \\ \operatorname{Im}(z)\geq 0}} e^{iaz} f(z)\,dz = 0$ for any $a > 0$

Th: Assume f has a finite number of singularities in $\operatorname{Im}(z) \geq 0$ and that $\displaystyle\lim_{R\to\infty} \int_{\substack{|z|=R \\ \operatorname{Im}(z)\geq 0}} f(z) e^{iaz}\,dz = 0$. for all $a > 0$

Assume f has no singularities in the real axis. Then $\displaystyle\int_{-\infty}^{\infty} f(x) e^{iax}\,dx = 2\pi i \sum \text{Res}$

for all $a>0$,
where the sum is over the residues of $f(z)e^{iaz}$ at the singularities of $f(z)$ in $\mathrm{Im}(z)>0$.

~~Example~~ : Obs

$$\int_{-\infty}^{\infty} f(x)\sin ax\,dx = \mathrm{Im}\int_{-\infty}^{\infty} f(x)e^{iax}\,dx$$

and
$$\int_{-\infty}^{\infty} f(x)\cos ax\,dx = \mathrm{Re}\left(\int_{-\infty}^{\infty} f(x)e^{iax}\,dx\right)$$

Ex: Compute $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin 2x}{x^2+1}\,dx$

$$\left|\frac{z}{z^2+1}\right| \underset{\text{if } |z|=R}{\leq} \frac{R}{R^2-1} \longrightarrow 0 \quad \text{as } R\to\infty$$

then last theorem applies

$$\int_{-\infty}^{\infty} \frac{x\sin 2x}{x^2+1}\,dx = \mathrm{Im}\left\{2\pi i\,\mathrm{Res}\left(\frac{z e^{i2z}}{z^2+1},\ i\right)\right\}$$

because i is the only singularity in $\operatorname{Im} z > 0$

$$\operatorname{Res}\left(\frac{z e^{i2z}}{z^2+1}, i\right) = \left.\frac{z e^{i2z}}{z+i}\right|_{z=i} = \frac{e^{-2}}{2}$$

$$\int_{-\infty}^{\infty} \frac{x \sin 2x}{x^2+1}\, dx = \pi e^{-2}$$

Th:

~~z_0^{ε} z_0 is the only singularity of f in $|z-z_0| \le \varepsilon$.~~

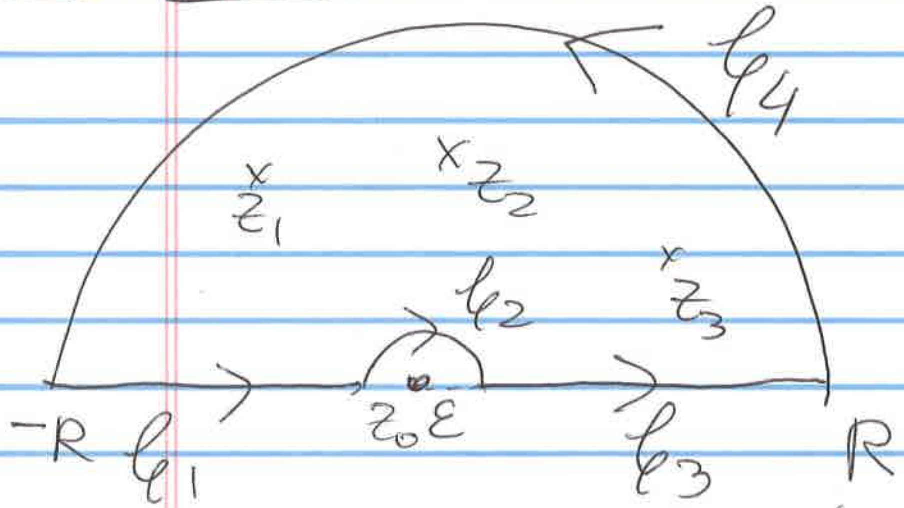
z_0 a singularity of f.

then

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} f(z)\, dz = -\pi i \operatorname{Res}(f, z_0)$$

C_ε : $|z-z_0| = \varepsilon$, $\operatorname{Im}(z-z_0) \geq 0$ and clockwise direction. (no proof given, but it is easy)

Obs:



1) Assume z_0 is the only singularity of f in the real axis

2) Assume $\lim_{R\to\infty} \int_{\ell_4} f(z)\,dz = 0$

3) Assume $z_1, \dots, z_n$ are the only singularities of f in $\operatorname{Im} z > 0$.

Then
$$\int_{-\infty}^{\infty} f(x)\,dx = i\pi \operatorname{Res}(f, z_0) + 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k)$$

<u>proof</u>:

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{C_1} f + \underbrace{\int_{C_2} f}_{\to -i\pi \operatorname{Res}(f, z_0)} + \int_{C_3} f + \underbrace{\int_{C_4} f}_{\to 0} = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k)$$

~~then the~~ $\lim\limits_{\substack{\varepsilon\to 0 \\ R\to\infty}} \int_{C_1} f + \int_{C_3} f = \int_{-\infty}^{\infty} f(x)\,dx$

then the result follows.

<u>Ex</u>: $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$

Compute $\displaystyle\int_0^\infty \frac{e^{ix}}{x}\,dx$ and take Im

$$\operatorname{Im}\int_0^\infty \frac{e^{ix}}{x}\,dx = \frac{1}{2}\operatorname{Im}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx$$

$$\lim_{R\to\infty} \int_{\substack{|z|=R \\ \operatorname{Im}(z)\geq 0}} \frac{e^{iz}}{z}\,dz = 0 \quad \text{(from today's lecture)}$$

(6)

$$\operatorname{Res}\left(\frac{e^{iz}}{z}, 0\right) = 1$$

then $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = i\pi \operatorname{Res}\left(\frac{e^{iz}}{z}, 0\right) = i\pi$

no singularities in $\operatorname{Im} z > 0$.

then $\displaystyle\int_{0}^{\infty} \frac{\sin x}{x}\,dx = \frac{1}{2}\operatorname{Im}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \frac{\pi}{2}$