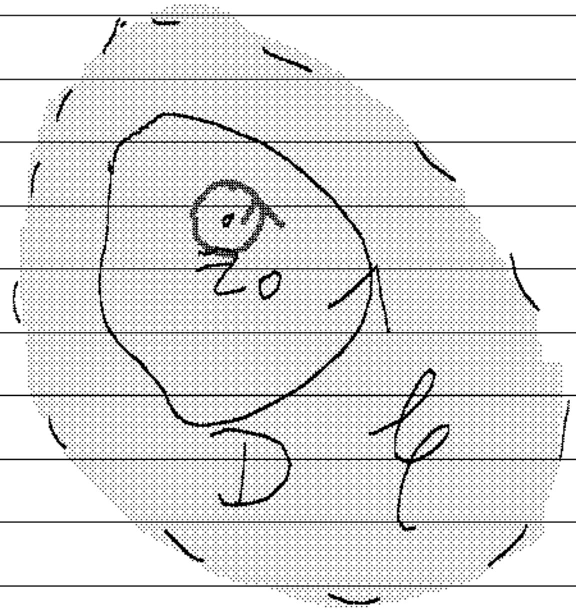
Note Title 7/20/2016

Cauchy's integral formula



D open simply connected. $\mathcal{C} \subset D$.
$\mathcal{C}$ simple closed curve. z_0 a point inside $\mathcal{C}$. f analytic in D. then

$$f(z_0) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{z - z_0}\, dz$$

proof: $g(z) = \dfrac{f(z)}{z - z_0}$ g is analytic in $D - \{z_0\}$

$$\oint_{\mathcal{C}} g = \oint_{|z-z_0|=\varepsilon} g = \int_0^{2\pi} g(z_0+\varepsilon e^{it})\,\varepsilon i e^{it}\,dt =$$

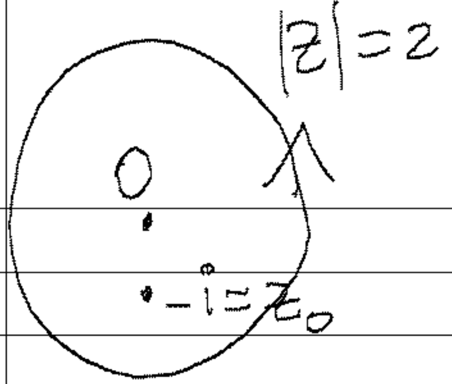
$dz = z'\,dt$

$$= \int_0^{2\pi} \frac{f(z_0+\varepsilon e^{it})}{\cancel{\varepsilon e^{it}}}\,\cancel{\varepsilon} i \cancel{e^{it}}\,dt \approx \int_0^{2\pi} f(z_0)\,i\,dt$$

$$= 2\pi i\, f(z_0) \qquad \varepsilon \to 0 \text{ then}$$

$$\frac{1}{2\pi i} \oint \frac{f(z)}{z-z_0}\,dz = f(z_0)$$

<u>Example:</u> $$\oint_{|z|=2} \frac{z^2-4z+4}{z+i}\,dz = \oint_{|z|=2} \frac{f(z)}{z-z_0}\,dz =$$



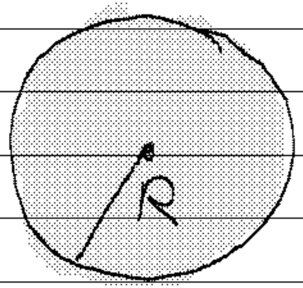
$f(z) = z^2 - 4z + 4$ $\qquad$ $= 2\pi i\, f(z_0) = 2\pi i\left((-i)^2 - 4(-i) + 4\right)$

$z_0 = -i$

$$\oint_{|z|=2} \frac{z^2 - 4z + 4}{z+i}\, dz = 2\pi i\,(3+4i)$$

__Power series__ $\displaystyle \sum_{k=0}^{\infty} a_k z^k = \lim_{N\to\infty} \sum_{k=0}^{N} a_k z^k$

__Facts:__ 1)



there exists $R \geq 0$ (R could be ∞) such that the above limit exists for all $|z|<R$, but it does not exists if $|z|>R$. R

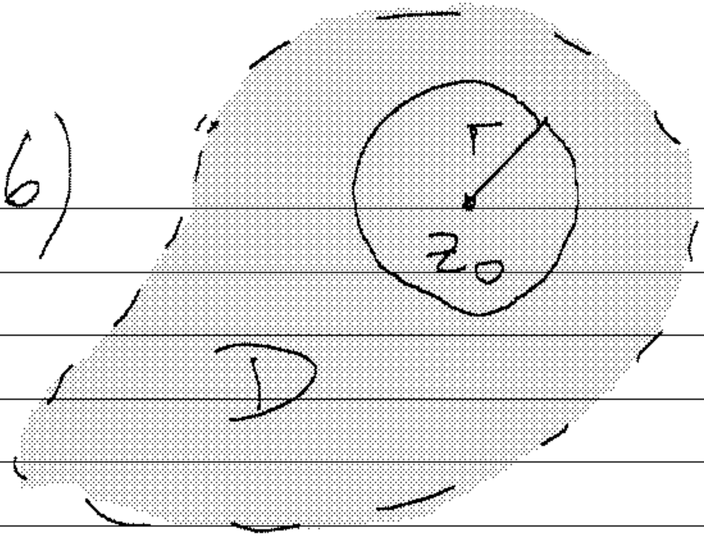
is called the radius of convergence

2) $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $\quad$ $f(z)$ is analytic for $|z| < R$

3) $a_k = \dfrac{f^{(k)}(0)}{k!}$

4) $f'(z) = \sum_{k=0}^{\infty} k\, a_k z^{k-1}$ the radius of convergence of this series is also R

5) $F(z) = \sum_{k=0}^{\infty} \dfrac{a_k}{k+1} z^{k+1}$. this series has radius of convergence R and $F'(z) = f(z)$

6) 

f analytic in D. $\{|z_0 - z| < r\} \subset D$

then

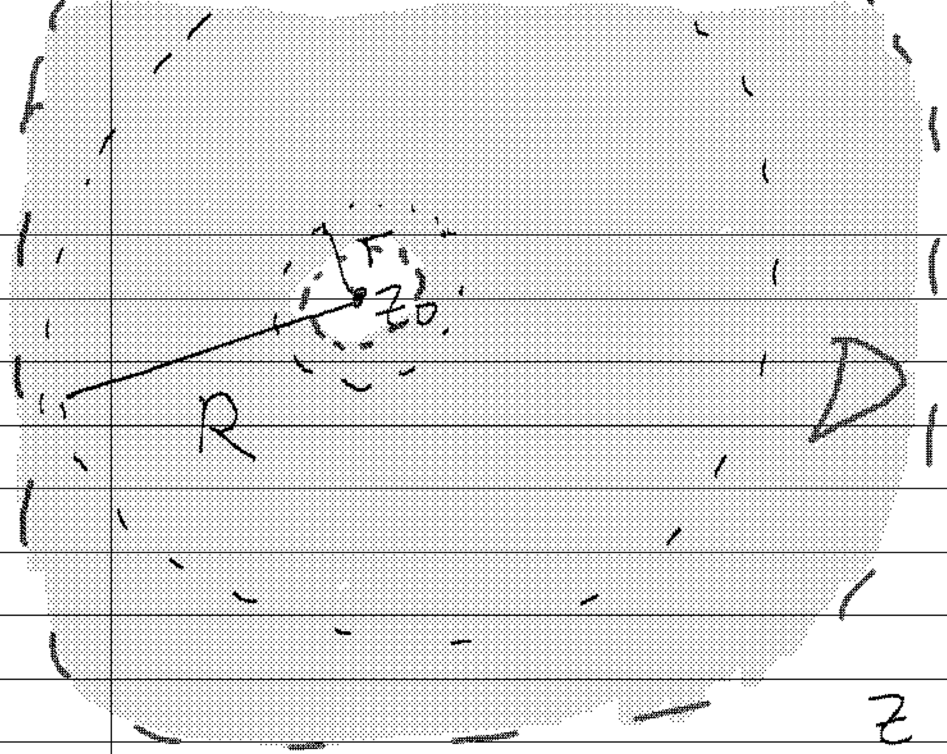
$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k, \quad \text{i.e.}$$

the radius of convergence of this series is at least r. this series is called the Taylor series of f at $z = z_0$

<u>Examples</u>: 1) $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$ for all $z \in \mathbb{C}$

2) $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$ for all $|z| < 1$

<u>Laurent series</u> $\{ r < |z - z_0| < R \} \subset D$



f analytic in D. f can be expanded as a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k \quad \text{for all}$$

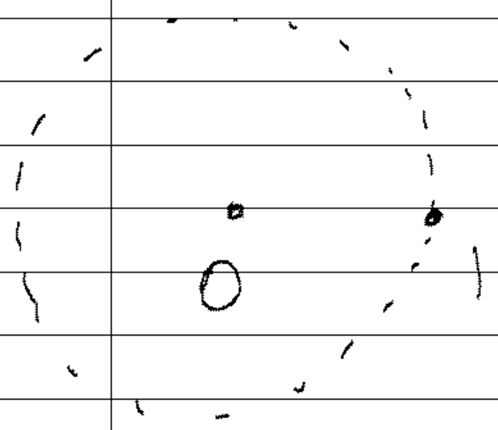
z such that $r < |z-z_0| < R$

<u>Example</u>: 1) $f(z) = \frac{1}{z}$ $\quad r=0 \quad R=\infty$

$f(z)$ is analytic for $0 < |z|$ $\qquad \frac{1}{z} = \sum_{k=-\infty}^{\infty} a_k z^k$

$$a_k = \begin{cases} 1 & \text{if } k=-1 \\ 0 & \text{otherwise} \end{cases}$$

2) $f(z) = \dfrac{1}{z(1-z)}$ Find the Laurent series of f at $z_0 = 0$



$r = 0$

$R = 1$

the series will converge for $0 < |z| < 1$

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \quad \text{then}$$

$$\frac{1}{z}\,\frac{1}{(1-z)} = \sum_{k=0}^{\infty} z^{k-1} = \sum_{n=-1}^{\infty} z^n$$

$k - 1 = n$

Obs:

z_0 inside $\mathcal{C}$ simple closed curve

$$\oint_{\mathcal{C}} (z-z_0)^k \, dz = \oint_{|z-z_0|=\varepsilon} (z-z_0)^k \, dz =$$

$$= \int_0^{2\pi} (\varepsilon e^{it})^k \, i\varepsilon e^{it} \, dt = i\,\varepsilon^{k+1} \int_0^{2\pi} e^{i(k+1)t} \, dt =$$

$$= \begin{cases} 2\pi i & k=-1 \\ \dfrac{i\varepsilon^{k+1}}{i(k+1)} e^{i(k+1)t}\Big|_0^{2\pi} \end{cases} = \begin{cases} 2\pi i & \text{if } k=-1 \\ 0 & \text{if } k\neq -1 \end{cases}$$

<u>Def</u>: $f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$

$a_{-1} = \mathrm{Res}(f, z_0)$ is the residue of f at z_0

Example : 1) $f(z) = \dfrac{1}{(1-z)z} = \dfrac{1}{z} + 1 + z + z^2 + \cdots$

$$\mathrm{Res}\left(\frac{1}{(1-z)z}, 0\right) = 1$$

2) $f(z) = \dfrac{1}{(1-z)\,z}$ at $z_0 = 1$

$$\frac{1}{z} = \frac{1}{1+(z-1)} = \sum_{k=0}^{\infty} \left[-(z-1)\right]^k = \sum_{k=0}^{\infty} (-1)^k (z-1)^k$$

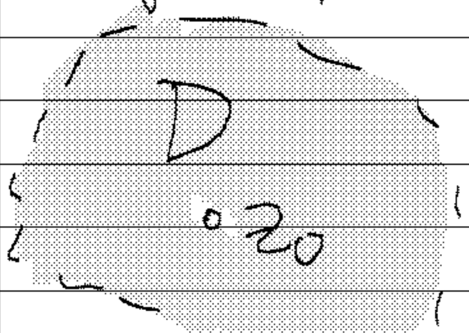
$$\frac{1}{1-w} = \sum_{k=0}^{\infty} w^k \qquad \frac{1}{z}\,\frac{1}{(1-z)} = \sum_{k=0}^{\infty} (-1)^{k-1} (z-1)^{k-1} =$$

$w = -(z-1)$

$$= \sum_{n=-1}^{\infty} (-1)^n (z-1)^n = \frac{(-1)}{z-1} + 1 - (z-1) + \dots$$

$$\operatorname{Res}\left(\frac{1}{z(1-z)}, 1\right) = -1$$

<u>Def</u>: f analytic around z_0 but maybe not defined at z_0.



We say that z_0 is a singularity. Let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \qquad \text{Laurent series}$$

Type of singularity	Laurent series	Example
Removable singularity	$a_0 + a_1(z-z_0) + \cdots$	$\frac{\sin z}{z}$
Pole of order n	$a_{-n}(z-z_0)^{-n} + \cdots\cdots$	$\frac{1}{(z-z_0)^n}$
Simple pole	$a_{-1}(z-z_0)^{-1} + \cdots\cdots$	
Essential singularity	$\cdots + \frac{a_{-n}}{(z-z_0)^n} + \cdots$	$e^{\frac{1}{z}}$ at $z_0 = 0$

<u>Def</u>: f analytic at z_0. f has a zero of order n at z_0 if $f(z) = a_n(z-z_0)^n + \cdots\cdots$ $a_n \neq 0$

Theorem: f & g analytic at z_0. f has a zero of order n at z_0. $g(z_0) \neq 0$. Let $F(z) = \dfrac{g(z)}{f(z)}$ has a pole of order n at z_0

Example: $F(z) = \dfrac{2z+5}{(z-1)(z+5)(z-2)^4}$

1 is a simple pole

-5 is a simple pole

2 is a pole of order 4

Simple pole at $z = z_0$

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \dots$$

$$(z - z_0) f(z) = a_{-1} + a_0(z - z_0) + \dots$$

$$\lim_{z \to z_0} (z - z_0) f(z) = a_{-1}$$

<u>theorem</u>: If f has a simple pole at z_0, then

$$\boxed{\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)}$$

$$f(z) = \frac{a_{-3}}{(z - z_0)^3} + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)} + \dots$$

$$(z-z_0)^3 f(z) = a_{-3} + a_{-2}(z-z_0) + a_{-1}(z-z_0)^2 + \dots$$

$$a_{-1} = \frac{1}{2!} \frac{d^2}{dz^2}\left[(z-z_0)^3 f(z)\right] \qquad z \to z_0$$

<u>theorem</u>: If f has a pole of order n at z_0, then

$$\lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right] = \text{Res}(f, z_0)$$

<u>Example</u>: $f(z) = \dfrac{1}{(z-1)^2 (z-3)}$

$$\operatorname{Res}(f,3) = \lim_{z\to 3} (z-3)f(z) = \lim_{z\to 3} \frac{\cancel{z-3}}{(z-1)^2\cancel{(z-3)}} = \frac{1}{4}$$

$$\operatorname{Res}(f,1) = \lim_{z\to 1} \frac{1}{1!}\frac{d}{dz}(z-1)^2 f(z) = \lim_{z\to 1} \frac{d}{dz}\frac{\cancel{(z-1)^2}}{\cancel{(z-1)^2}(z-3)} =$$

$$\lim_{z\to 1} \frac{d}{dz}\frac{1}{(z-3)} = \lim_{z\to 1} \frac{-1}{(z-3)^2} = -\frac{1}{4}$$