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<u>Differential Equations</u>

<u>Def</u>: Unkown is a function. The function and some of its derivatives appear in the equations. If the unknown is a function of one variable, the equation is called an ordinary differential equation (ode)

Example: $y'' + 3xy = \cos x$ (1)

Goal; Find $y = y(x)$ that satisfies

Def: The order of an ode is the order of the highest derivative in the equation

Example: The ode (1) is of order 2.

Def: An ode is said to be linear if it is of the form

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \dots\dots + a_1(x)\, y' + a_0(x)\, y = g(x)$$

$g, a_0, a_1, \dots, a_n$ are known functions of x

Notation $y^{(n)} = \dfrac{d^n y}{dx^n}$

Examples: a) $3x y'' + \cos x \; y = e^x$ is linear

b) $(y')^2 + x^2 = 7 \cos y$ is non-linear

Obs: $y' = y \quad (2)$

$y = e^x$ and $y = 7e^x$ are both solutions of equation (2). Most odes have many solutions.

Example $(*)\begin{cases} x' = 50 \\ x(0) = 5 \end{cases}$ $\qquad x = x(t)$

then $x(t) = 5 + 50t$. There is only one solution to

(*) (both equations simultaneously).

IVP, Initial-value problem.

$$\left.\begin{aligned} & y^{(n)} = F(x, y, y', \ldots, y^{(n-1)}) \\ & y(x_0) = y_0 \\ & y'(x_0) = y_1 \\ & \vdots \\ & y^{(n-1)}(x_0) = y_{n-1} \end{aligned}\right.$$

ode, n^{th} order

initial values (n of them)

$y_0, \ldots, y_{n-1}$ are known numbers.

<u>Fact</u>: In general, there will be only one solution to these IVPs

Examples: 1) First order IVP: $\frac{dy}{dx} = f(x,y)$ & $y(x_0) = y_0$

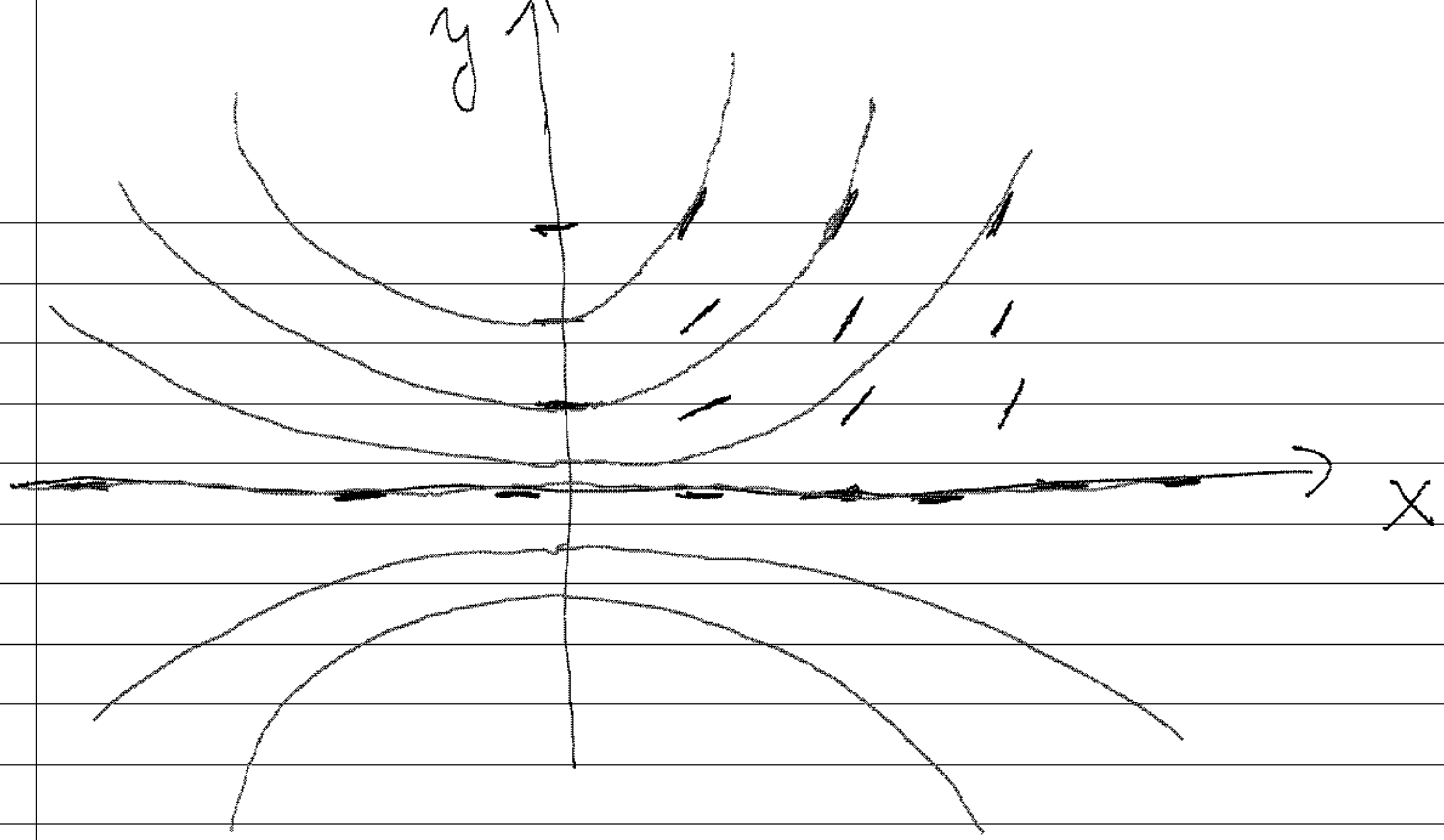
2) Second order IVP: $y'' = f(x,y,y')$ & $y(x_0) = y_0$
$y'(x_0) = y_1$

Understanding the behavior of the solutions without solving the odes

Direction fields: This is for first order equations

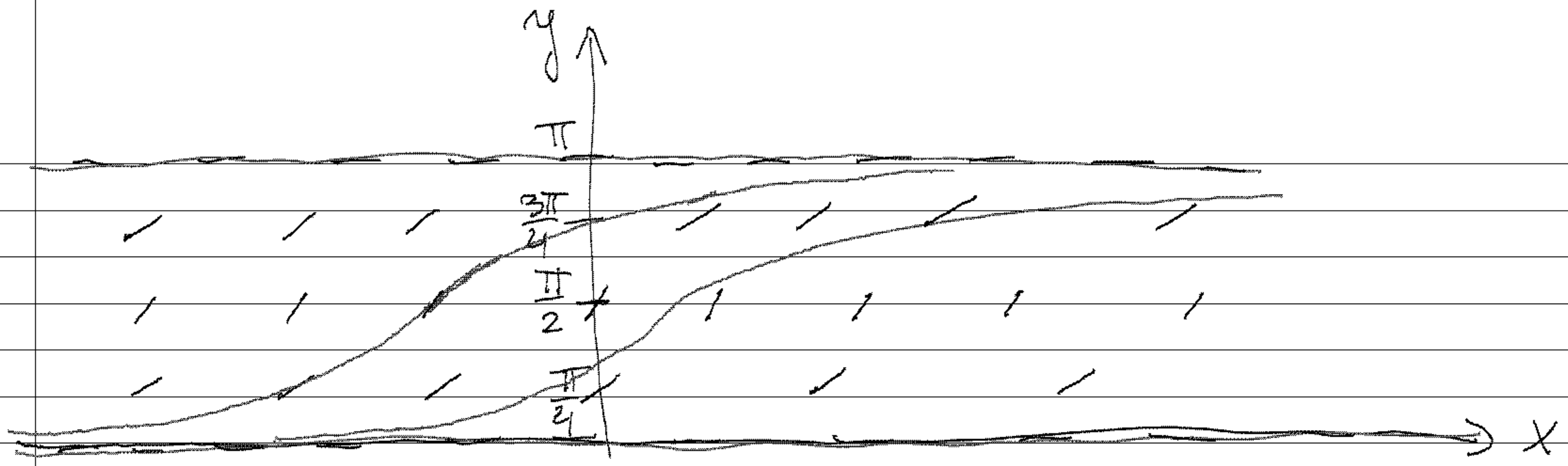
Example 1) $y' = \frac{1}{2}xy$

Draw little segments in the x-y plane with slope $\frac{1}{2}xy$



$$y' = \frac{1}{2}xy$$

Example $\quad y' = \sin y$



<u>Autonomous odes</u>: These are odes where the independent variable does not appear explicitly in the ode.

$$y^{(n)} = F(y^{(n-1)}, \ldots, y', y)$$

<u>Example</u>: 1) $y' = 1+y^2$ is 1^{st} order autonomous

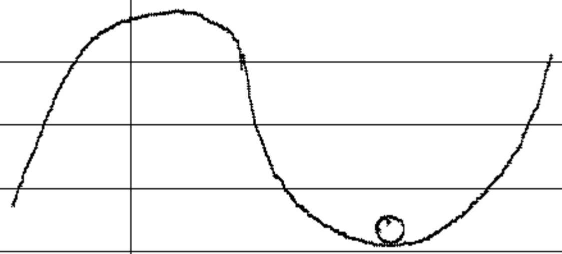
2) $y' = xy$ is not autonomous

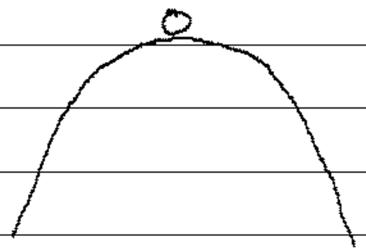
<u>Notation</u>: $\dot{x} = f(x)$ $\quad \dot{x} = x' = \frac{dx}{dt}$ $\quad t =$ independent variable

<u>Def</u>: We say x_0 is a fixed point or a critical point of $\dot{x} = f(x)$ if $f(x_0) = 0$

<u>Obs</u>: Let x_0 be a fixed point of $\dot{x} = f(x)$. Then $x(t) = x_0$ for all t is a solution of $\dot{x} = f(x)$ because $\dot{x} = 0$ and $f(x) = 0$

<u>Def</u>: Fixed points are also called equilibrium points.





stable equilibrium unstable equilibrium

Def: Let x_0 be a fixed point of $\dot{x} = f(x)$. We say x_0 is stable if $x(t)$ remains close to x_0 as long as $x(0)$ was close to x_0. Otherwise, we say x_0 is unstable.

Solving first order autonomous odes

$$\frac{dx}{dt} = f(x) \longrightarrow \frac{dx}{f(x)} = dt \longrightarrow \int \frac{dx}{f(x)} = \int dt$$

then try to solve for x.

Examples: 1) $\dot{x} = kx$ $\quad k$ constant

$$\int \frac{dx}{x} = \int k\,dt \longrightarrow \ln|x| = kt + c \longrightarrow |x| = e^c e^{kt}$$

$$\boxed{\begin{array}{l} A = e^c \text{ if } x>0 \\ A = -e^c \text{ if } x \le 0 \end{array}} \quad x = A e^{kt} \leftarrow \text{these are all the solutions}$$

$$A \in \mathbb{R}$$

$(*)$

2) $\dot{x} = x(1-x) \longrightarrow \displaystyle\int \frac{dx}{x(1-x)} = \int dt \longrightarrow$

$$\frac{1}{x(1-x)} = \frac{a}{x} + \frac{b}{(x-1)} = \frac{a(x-1)+bx}{x(x-1)} = \frac{(a+b)x - a}{x(x-1)}$$

$a+b=0 \qquad -a=-1 \qquad a=1 \quad = -1$

$$\int \frac{dx}{x(1-x)} = \int \frac{dx}{x} - \int \frac{dx}{x-1} = \ln|x| - \ln|x-1| \overset{\text{from } \circledast}{=} t + c$$

$$\ln\left|\frac{x}{x-1}\right| = t + C \quad \longrightarrow \quad \left|\frac{x}{x-1}\right| = e^{C} e^{t}$$

$$\frac{x}{x-1} = A e^{t} \qquad A \in \mathbb{R}$$

$$x = x A e^{t} - A e^{t}$$

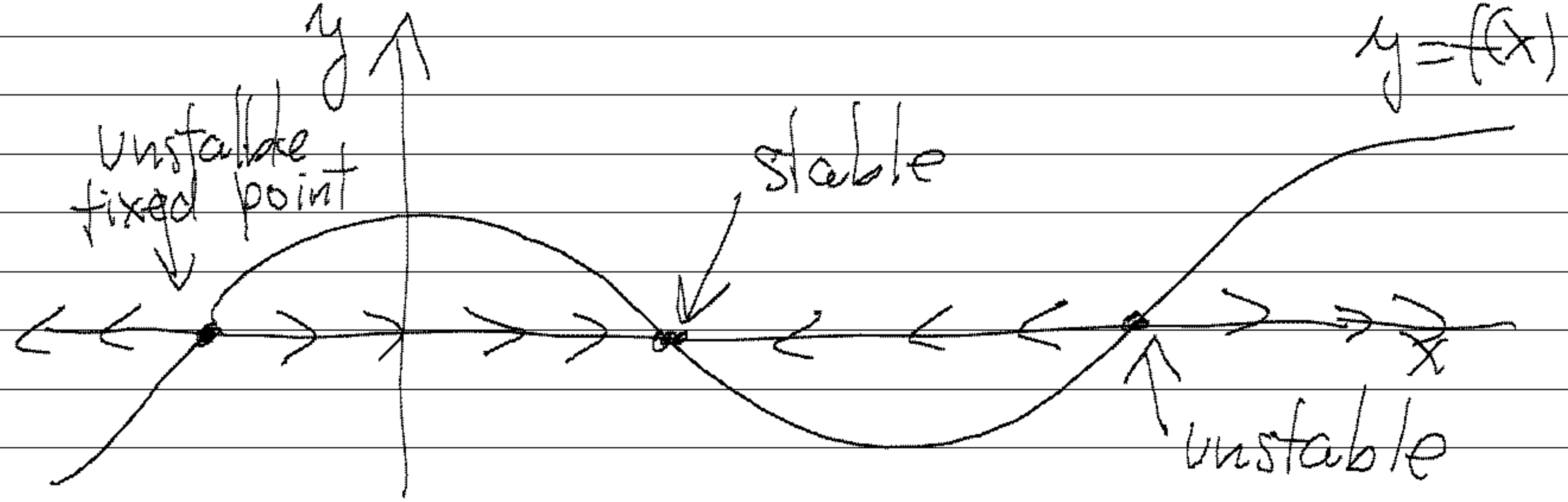
$$x(1 - Ae^{t}) = -A e^{t}$$

$$\boxed{x = -\frac{A}{(1-Ae^{t})} e^{t}} \quad \text{or} \quad \boxed{x = \frac{1}{1 - \frac{1}{A} e^{-t}}}$$

Phase line: 1st order autonomous odes

$\dot{x} = f(x)$

Plot $y = f(x)$

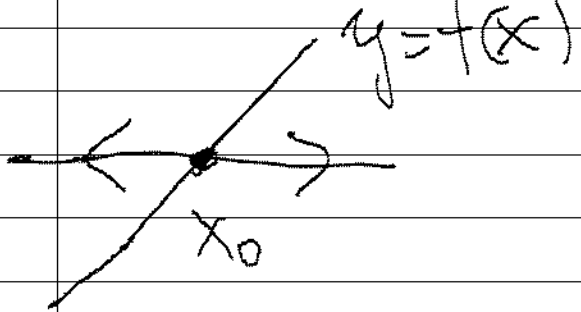


the zeros of $f(x)$ are the fixed points

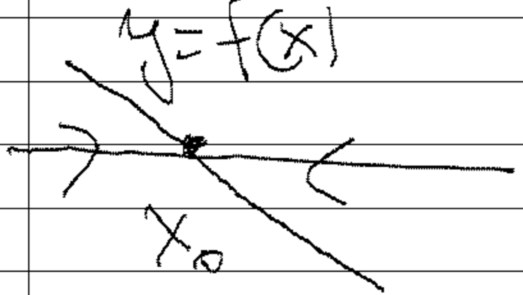
draw arrows in the x-axis, to the rigth if $f(x) > 0$, to the left if $f(x) < 0$

The arrows indicate the direction of motion

Obs: let x_0 be a fixed point of $\dot{x} = f(x)$



If $f'(x_0) > 0$ then x_0 is unstable



If $f'(x_0) < 0$ then x_0 is stable

If $f'(x_0) = 0$, we do not know

Separable equations $\qquad \dfrac{dy}{dx} = g(x)\, h(y)$

Separation of variables

$$\int \frac{dy}{h(y)} = \int g(x)\, dx$$ then try to solve for y

Example: $\frac{dy}{dx} = -\frac{x}{y}$ $\quad y(4) = 3$

$$\int y\, dy = -\int x\, dx \qquad \frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$y = \pm\sqrt{2C - x^2} \qquad y(4) = 3 \Rightarrow$$

$$3 = \sqrt{2C - 16} \Rightarrow \boxed{y = \sqrt{25 - x^2}}$$

Linear first order differential equations

(1) $$y' + P(x)\, y = f(x)$$

$$I(x) = e^{\int P(x) dx}$$ then $$I'(x) = P(x)\, e^{\int P(x) dx}$$

$$I'(x) = P(x)\, I(x)$$

Multiply eq (1) by $I(x)$

$$I y' + \underbrace{I P}_{= I'} y = I f$$

$$I y' + I' y = I f$$

$$\int (I y)' = \int I f$$

$$I y = \int I f$$

$$y = \frac{1}{I(x)} \int I(x) f(x)\, dx$$

Example 1) $y' - 3y = 6$ $\qquad y' + Py = f$

$P = -3 \qquad I = e^{\int P}$

$I = e^{-3x}$

$$e^{-3x} y' - 3 e^{-3x} y = 6 e^{-3x}$$

$$\frac{d}{dx}\left(e^{-3x} y\right) = 6 e^{-3x}$$

$$e^{-3x} y = \int 6 e^{-3x}$$

$$e^{-3x} y = -2 e^{-3x} + C$$

$$\boxed{y = -2 + C e^{3x}}$$