MATH 6635 HW2 Solution:

Problem $\boxed{5}$ $X \sim N(\mu, \sigma^2) \Rightarrow \underset{\text{pdf}}{f(x)} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$ let $y = \frac{x-\mu}{\sigma}$ then $dx = \sigma dy$

$= \int_{-\infty}^{\infty} (\sigma y + \mu) \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2}} \sigma dy$

$= \sigma \int_{-\infty}^{\infty} y \cdot \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy}_{\sim N(0,1)} + \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$

$= \sigma \cdot 0 + \mu \cdot 1$

$= \mu$

$E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} (\sigma y + \mu)^2 \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2}} \cdot \sigma dy$

$= \sigma^2 \underbrace{\int_{-\infty}^{\infty} y^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy}_{E[X^2]=1-0=1} + 2\sigma\mu \int_{-\infty}^{\infty} \frac{y}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$ $\sim N(0,1)$

$+ \mu^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$

$= \sigma^2 \cdot 1 + 2\sigma\mu \cdot 0 + \mu^2 \cdot 1$

$= \sigma^2 + \mu^2$

$Var[X] = E[X^2] - E[X]^2 = \sigma^2 + \mu^2 - \mu^2$

$= \sigma^2$

Problem $\boxed{6}$ $X \sim N(\mu, \sigma^2)$ Let $Y = aX + b$

$P(aX + b \le y) = P(X \le \frac{y-b}{a}) = \int_{-\infty}^{\frac{y-b}{a}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$ Let $x = \frac{y-b}{a}$

$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\frac{y-b}{a}-\mu)^2}{2\sigma^2}} \cdot \frac{1}{a} dy$

$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}a\sigma} e^{-\frac{(y-a\mu-b)^2}{2a^2\sigma^2}} dy$ $\therefore \mu = a\mu + b$

$\sigma = a^2\sigma^2$

$\therefore Y \sim N(a\mu + b, a^2\sigma^2)$

problem $\boxed{7}$: $X \sim N(\mu, \sigma^2)$ $Z = e^X$

$$E[Z] = E[e^X] = \int_{-\infty}^{\infty} e^x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$ Let $y = x - \mu$. $dy = dx$.

$$= \int_{-\infty}^{\infty} e^{y+\mu} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} dy$$

$$= e^{\mu} \int_{-\infty}^{\infty} e^{y} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} dy$$

$$= e^{\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-y^2 + 2\sigma^2 y}{2\sigma^2}} dy$$

$$= e^{\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y-\sigma^2)^2 + \sigma^4}{2\sigma^2}} dy$$

$$= e^{\mu + \frac{\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\sigma^2)^2}{2\sigma^2}} dy$$

$$= e^{\mu + \frac{\sigma^2}{2}}$$

$$E[Z^2] = E[e^{2X}] = \int_{-\infty}^{\infty} e^{2x} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$ Let $y = x - \mu$ $dy = dx$

$$= \int_{-\infty}^{\infty} e^{2(\mu+y)} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} dy$$

$$= e^{2\mu} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-y^2 + 4y\sigma^2}{2\sigma^2}} dy.$$

$$= e^{2\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y-2\sigma^2)^2 + 4\sigma^4}{2\sigma^2}} dy.$$

$$= e^{2\mu + 2\sigma^2}$$

$$Var[Z] = E[Z^2] - E[Z]^2 = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$$

problem $\boxed{8}$: $X \sim N(\mu_X, \sigma_X^2)$. $Y \sim N(\mu_Y, \sigma_Y^2)$.

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(z-x) f_X(x) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(z-x-\mu_Y)^2}{2\sigma_Y^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma_X} \cdot e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{\sigma_X^2+\sigma_Y^2}} \cdot e^{-\frac{[z-(\mu_X+\mu_Y)]^2}{2(\sigma_X^2+\sigma_Y^2)}} \cdot \frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sqrt{\sigma_X^2+\sigma_Y^2}}} \cdot e^{-\frac{\left(x - \frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_X^2+\sigma_Y^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sqrt{\sigma_X^2+\sigma_Y^2}}\right)^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_X^2+\sigma_Y^2}} e^{-\frac{(z-(\mu_X+\mu_Y))^2}{2(\sigma_X^2+\sigma_Y^2)}} \cdot 1.$$

$\therefore X+Y \sim N(\mu_X+\mu_Y, \sigma_X^2+\sigma_Y^2)$.