Homework

Problem 1 (5 points, due on 1/14): Let V be the price of an American call option. Let K be the strike and S the spot price of the asset. Show that the no arbitrage principle implies that $V \geq S - K$.

Problem 2 (2 points, due on 1/14): Out of students in a class, 60% like chocolate, 70% strawberries, and 40% both. Determine the probability that a randomly selected student does not like neither chocolate, nor strawberries.

Problem 3 (2 points, due on 1/14): We are given three coins: one has heads on both faces, the second has tails in both faces and the third has a tail in one face and a head in the other. We select a coin at random, toss it, and it comes heads. What is the probability the other face is tails?

Problem 4 (5 points, due 1/14): Let V be the price of an European put option at time t. Let K be the strike and $S = S(t)$ the price of the asset at time t. Let T be the exercise time. Show that the no arbitrage principle implies that $V \geq Ke^{-r(T-t)} - S$.

Problem 5 (3 points, due on 1/21): $X \sim \mathcal{N}(\mu, \sigma^2)$. Show that $\mathrm{E}[X] = \mu$ and $\mathrm{Var}X = \sigma^2$.

Problem 6 (2 points, due on 1/21): $X \sim \mathcal{N}(\mu, \sigma^2)$. Show that $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

Problem 7 (3 points, due on 1/21): If $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Z = e^X$, show that $\mathrm{E}[X] = e^{\mu + \frac{\sigma^2}{2}}$ and $\mathrm{Var}X = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$.

Problem 8 (2 points, due on 1/21): $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$. Show that, if X and Y are independent, then $X + Y \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

Problem 9 (3 points, due on 1/28): W Wiener process. $X = W$. $F(X) = X^2$. a) Compute dF. b) Let $T > 0$. Compute the expected value of $W^2(T) - W^2(0)$.

Problem 10 (3 points, due on 1/28): W Wiener process. $dS = \mu S dt + \sigma S dW$. Find the stochastic differential equation satisfied by S^n, where n is a positive integer.

Problem 11 (10 points, due on 2/4): Using the binomial method, find f, the price of an European call at time $t = 0$, with $S(0) = K = 60$, $r = 0.08$, $\sigma = 0.4$, and maturity in five months. Plot the value of f obtained as a function of N, for $1 \leq N \leq 50$. Also report the values obtained for $N = 100$, $N = 200$ and $N = 400$.

Problem 12 (10 points, due on 2/4): Using the binomial method, find f, the price of an American put at time $t = 0$, with $S(0) = K = 50$, $r = 0.1$, $\sigma = 0.4$, and maturity in five months. Plot the value of f obtained as a function of N, for $1 \leq N \leq 50$. Also report the values obtained for $N = 100$, $N = 200$ and $N = 400$.

Problem 13 (3 points, due on 2/11): Let $A \in \mathbb{R}^{k \times n}$. Show that

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{k} |a_{ij}|.$$

Problem 14 (3 points, due on 2/11): Let $A \in \mathbb{R}^{n \times n}$. Let λ and β be the largest and smallets eigenvalues of A in absolute value. Show that

$$\kappa(A) \geq \frac{|\lambda|}{|\beta|},$$

where $\kappa(A)$ is the condition number of A.

Problem 15 (10 points, due on 2/11): Write a code to solve linear systems by Gaussian Elimination and solve the system $Ax = b$, where

$$A = \begin{bmatrix} 0 & 1 & -2 & 1 & 3 \\ 2 & 5 & -3 & 4 & 7 \\ -1 & 0 & 3 & -2 & 4 \\ 1 & 3 & 8 & -2 & 0 \\ 0 & 0 & -5 & 4 & 2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \\ 0 \end{bmatrix}.$$

Problem 16 (10 points, due on 2/25): Write a code to compute the Cholesky factorization. Use to compute the Cholesky factorization of

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 5 & -2 & 8 \\ 0 & -2 & 2 & -7 \\ 2 & 8 & -7 & 38 \end{bmatrix}.$$

Then use this factorization to solve $Ax = b$, with

$$b = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 0 \end{bmatrix}.$$

Problem 17 (10 points, due on 2/25): Solve $y''(x) = -1$ with bounday conditions $y(0) = y(1) = 0$ in the interval $[0, 1]$ numerically using the finite difference discretization described in class. Use the Cholesky factorization to solve the resulting linear system. Use $N = 4, 8, 16, 32, 64, 128$ and plot the solutions obtained in the same graph. Also plot in that graph the exact solution $y = -x(x-1)/2$.

Problem 18 (3 points, due on 2/25): Let $A \in \mathbb{R}^{n \times n}$. Let D be the diagonal of A. Show that

$$\|I - D^{-1}A\|_\infty = \max_{1 \leq i \leq n} \sum_{j \neq i} \frac{|a_{ij}|}{|a_{ii}|}.$$

Problem 19 (10 points, due on 2/25): (a) Write a code to solve linear systems by the Jacobi method and solve the system $Ax = b$, where

$$A = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Start with $x^{(0)} = 0$. Plot $x_i^{(k)}$ vs k for each i, all in the same graph. Restrict youself to $0 \le i \le 10$. What is the exact solution?

(b) Repeat (a) but using Gauss-Seidel.

Problem 20 (10 points, due on 3/11): Let

$$f(x) = \frac{1}{1+x^2}.$$

Let $P_n(x)$ be the polinomial of degree n that interpolate $f(x)$ in the interval $[-3, 3]$ at $n+1$ equidistant points including the boundary points, i.e. at $x_i = -3 + i6/n$, with $0 \le i \le n$. Let $Q_n(x)$ be the polinomial of degree n that interpolate $f(x)$ in the interval $[-3, 3]$ at $n+1$ Tchebicheff points. Plot in the same graph f, P_n and Q_n for $n = 2, 4, 8, 16$ (use colors or dashed and solid lines to make it easy to see). Use the methods described in class to compute $P_n(x)$ and $Q_n(x)$, i.e. write the program to compute $y[x_0, \ldots, x_i]$ and then the program to evaluate the polinomials fast.

Problem 21 (10 points, due on 3/11): Let

$$f(x) = \cos(x).$$

Let $S_n(x)$ be the spline of degree 3 that interpolate $f(x)$ in the interval $[0, 2\pi]$ at $n+1$ equidistant points including the boundary points, i.e. at $x_i = i2\pi/n$, with $0 \le i \le n$. Let $R_n(x)$ be the spline of degree 1 that interpolate $f(x)$ in the interval $[0, 2\pi]$ at the same points. Plot in the same graph f, S_n and R_n for $n = 4, 8, 16$ (use colors or dashed and solid lines to make it easy to see). Use the methods described in class to compute $S_n(x)$ and $R_n(x)$. Explain what you do, i.e. the linear system you solve to get the splines, what method you use to solve that linear system, etc.

Problem 22 (5 points, due on 3/11): (a) Write a code to use the bisection method. Use that code to find a solution of

$$\left(\frac{x}{2}\right)^2 - \sin x = 0.$$

Start with $a_0 = 1$ and $b_0 = 2$. Plot a_i vs i and b_i vs i and $x_i = (a_i + b_i)/2$ vs i, all in the same graph, where i is the number of iteration and a_i and b_i are the left and right end points of the interval containing the root after the i^{th} iteration. Plot the range $0 \le i \le 10$. Make a table with $x_i = (a_i + b_i)/2$ vs i for $0 \le i \le 10$. Find the root with 6 digits of precision. How many iterations were necessary?

(b) Write a code to use the Newton's method. Use that code to find a solution of

$$\left(\frac{x}{2}\right)^2 - \sin x = 0.$$

Start with $x_0 = 2$. Plot x_i vs i, where i is the number of iteration, in the same graph as the plots with the bisection method. Plot the range $0 \le i \le 10$. Make a table with x_i vs i for $0 \le i \le 10$. Find the root with 6 digits of precision. How many iterations were necessary?

Problem 23 (10 points, due on 4/1): (a) Write a code to use the trapeziodal method to compute the integral $\int_{-1}^{1} -x^3 + x + 1 dx$. Make a table with the results obtain with $h = 1/2^i$ $1 \le i \le 8$. What is the exact value of the integral?

(b) Write a code to use Richardson extrapolation on the results obtained in (a). Make a *triangular* table like in class with the results obtained

(c) Write a code to use the Simpson method to compute the integral $\int_{-1}^{1} -x^3 + x + 1 dx$. Make a table with the results obtain with $h = 1/2^i$ $1 \le i \le 4$. Comment on the rsults obtained

Problem 24 (10 points, due on 4/1): An European call option is \$2.85. The strike price \$54. The expiry time is five months. The asset current price is \$50. The risk-free interest rate is 7%. What is the implied volatility?. Proceed as explained in class, i.e. use the Black-Scholes formula, and you will have to use bisection or Newton-Rapson's, and also some way to integrate like trapeziodal. Explain your steps.

Problem 25 (10 points, due on 4/8): Same as problem 23 but $\int_{-1}^{1}(1 - x^2) + \cos(\pi x/2) dx$.

Problem 26 (10 points, due on 4/8): Consider the initial value problem $y' = \cos(x)y$, $y(0) = 1$. Solve this problem numerically for $0 \le x \le 10$. Use any method of your prefernce but explain the details of your work, i.e. method work, h, distance between node points, etc. Plot your results. Also obtain the solution analytically and plot it in the same figure.

Problem 27 (10 points, due on 4/8): Same as problem 26, but for the initial value problem $y'' + xy = \cos x$ with $y(0) = y'(0) = 0$. Solve for $0 \le x \le 10$. Do not solve analitically. Use a different method that you used for problem 26

Problem 28 (20 points, due on 4/15): Solve numerically the heat equation $u_t = u_{xx}$ for $0 \le x \le 1$ and $t \ge 0$. Use the initial conditions $u(x, 0) = 1$ and the boundary conditions $u(0, t) = 2$ and $u(1, t) = 0$ for $t > 0$. Plot u vs x (the u obtained numerically), for the following values of t: $t = 0$, $t = 0.1$, $t = 0.5$, $t = 1$ and $t = 10$, all in the same graph. Repeat as follows:

1) Use method 1 with $h = 0.1$, $k = 0.004$
2) Use method 1 with $h = 0.1$, $k = 0.01$
3) Use method 1 with $h = 0.01$, $k = 0.00004$
4) Use method 1 with $h = 0.01$, $k = 0.0001$
5) Use Crank-Nicolson with $h = 0.1$, $k = 0.1$

6) Use Crank-Nicolson with $h = 0.01$, $k = 0.01$
So you should turn in 6 figures. Comment on the results obtained.