Note Title 4/8/2015

$\tilde{U}_i^n = U_i^n + z_i^n$ Method 1

we compute — we want — error

z_i^n satisfies

(*) $z_j^{n+1} = z_j^n + \frac{k}{h^2}\left(z_{j-1}^n - 2z_j^n + z_{j+1}^n\right)$

$z_0^n = z_M^n = 0$

$z_j^0 = \delta(x_j)$

<u>Fact:</u> No matter what $z_j^0 = \delta(x_j)$ is, there are constants $\gamma_1, \ldots, \gamma_{M-1}$ such that

$$z_j^0 = \sum_{l=1}^{M-1} \gamma_l \sin\left(\frac{l\pi}{M} j\right)$$

Thus, we can assume $z_j^0 = \sin\left(\frac{l\pi}{M} j\right)$, l fixed, $1 \le l \le M-1$.
And we want to understand when these $M-1$ solutions, one for each l, remains bounded. This will be the parameter regime where the method is stable.

Set $z_j^n = H^n \sin\left(\frac{l\pi}{M} j\right)$ Plug into equation (*)

$$z_j^{n+1} = z_j^n + \frac{k}{h^2}\left(z_{j-1}^n - 2z_j^n + z_{j+1}^n\right)$$

$$H^{n+1} \sin\left(\frac{l\pi}{M} j\right) = H^n \sin\left(\frac{l\pi}{M} j\right) + \frac{k}{h^2}\left(H^n \sin\left(\frac{l\pi}{M}(j-1)\right) - 2H^n \sin\left(\frac{\pi l}{M} j\right) + \right.$$

$$\left. + H^n \sin\left(\frac{l\pi}{M}(j+1)\right)\right]$$

(1) $\sin\left(\frac{l\pi}{M}(j-1)\right) = \sin \frac{l\pi j}{M} \cos \frac{l\pi}{M} - \cos \frac{l\pi j}{M} \sin \frac{l\pi}{M}$

$\sin(a+b) = \sin a \cos b + \cos a \sin b$ trigonometric identity

(2) $\sin\left(\frac{l\pi}{M}(j+1)\right) = \sin \frac{l\pi j}{M} \cos \frac{l\pi}{M} + \cos \frac{l\pi j}{M} \sin \frac{l\pi}{M}$

(1)+(2) $= 2 \sin \frac{l\pi j}{M} \cos \frac{l\pi}{M}$

$$H \sin\left(\frac{l\pi}{M} j\right) = \sin\left(\frac{l\pi}{M} j\right) + \frac{k}{h^2} 2 \sin\left(\frac{l\pi j}{M}\right)\left(\cos\left(\frac{l\pi}{M}\right) - 1\right)$$

$$H = 1 + \frac{2k}{h^2}\left(\cos\left(\frac{l\pi}{M}\right) - 1\right)$$

For stability, we want

$$-1 \leq 1 + \frac{2k}{h^2}\left(\cos\left(\frac{l\pi}{M}\right) - 1\right) \leq 1$$

$$-2 \leq \frac{2k}{h^2}\left(\cos\left(\frac{l\pi}{M}\right) - 1\right) \leq 0$$

$$\frac{k}{h^2} \leq \frac{1}{1 - \cos\left(\frac{l\pi}{M}\right)} \qquad \frac{k}{h^2} \leq \frac{1}{1 - \cos\left(\frac{(M-1)\pi}{M}\right)} \approx \frac{1}{2}$$

$\boxed{k \le \frac{h^2}{2}}$ Method 1 is stable only for $k \le \frac{h^2}{2}$

<u>Fact</u>: Method 1 converges in this parameter regime, i.e. if $u(x,t)$ is the exact solution, then

$$\lim_{\substack{h \to 0 \\ k \le \frac{h^2}{2} \\ nk = t,\ jh = x}} V_j^n = u(x,t)$$

Stability for Crank-Nicolson

Again $z_j^n = H^n \sin\left(\frac{\pi l}{M} j\right)$

$$\frac{Z_j^{n+1} - Z_j^n}{k} = \frac{1}{2h^2}\left(Z_{j-1}^n - 2Z_j^n + Z_{j+1}^n + Z_{j-1}^{n+1} - 2Z_j^{n+1} + Z_{j+1}^{n+1}\right)$$

$$\frac{H-1}{k} = \frac{1}{h^2}\left(\left(\cos\left(\frac{\pi l}{M}\right) - 1\right) + H\left(\cos\left(\frac{\pi l}{M}\right) - 1\right)\right)$$

Multiply by k

$$H\left(1 + \frac{k}{h^2}\left(1 - \cos\left(\frac{\pi l}{M}\right)\right)\right) = 1 + \frac{k}{h^2}\left(\cos\left(\frac{\pi l}{M}\right) - 1\right)$$

$$H = \frac{1 - \frac{k}{h^2}\left(1 - \cos\left(\frac{\pi l}{M}\right)\right)}{1 + \frac{k}{h^2}\left(1 - \cos\left(\frac{\pi l}{M}\right)\right)}$$

$|H| < 1$ no matter k or h

CN is always stable. It also converges as $h \to 0$, $k \to 0$

Take $k = O(h)$

What if we want to solve $u_t = D\, u_{xx}$ with $D \neq 1$

D constant $\quad y = \frac{x}{\sqrt{D}} \quad \frac{\partial}{\partial x} = \frac{1}{\sqrt{D}} \frac{\partial}{\partial y} \Rightarrow \frac{\partial^2}{\partial x^2} = \frac{1}{D} \frac{\partial^2}{\partial y^2}$

thus

$$u_t = u_{yy}$$

In method 1, $\quad \Delta t \le \frac{(\Delta y)^2}{2} = \frac{(\Delta x)^2}{2D} \qquad k \le \frac{h^2}{2D}$

In CN $\quad \Delta t = O(\Delta y) = \frac{O(\Delta x)}{\sqrt{D}}$

Black Scholes equation

$S(t)$ = asset price at time t

$f = f(S,t)$ = option value at time t if asset value at time t is S

r = risk free interest

σ = volatility

PDE $\quad \dfrac{\partial f}{\partial t} + rS\dfrac{\partial f}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 f}{\partial S^2} = rf$

T = option maturity time

We are interested in $\quad 0 \le t \le T$

Grid points $S_i = i\Delta S$ $\quad 0 \leq i \leq M$

$t_j = j\Delta t$ $\quad 0 \leq j \leq N$

$\Delta t = \frac{T}{N}$ $\quad M\Delta S = S_{max}$ (we can not discretize all the way to $S = +\infty$)

$f_{ij} \simeq f(S_i, t_j)$

Obs: No initial conditions. Insted, we have terminal conditions

For a call $f(S,T) = \max\{S-K, 0\}$ $\quad 0 \leq S \leq S_{max}$

For a put $f(S,T) = \max\{K-S, 0\}$ $\quad 0 \le S \le S_{max}$

On boundary conditions

$\boxed{\text{European put}}$

S very large, "then option likely to be worthless",

$\boxed{f(S_{max}, t) = 0}$

at $S=0$ $\quad \frac{\partial f}{\partial t} = rf$ $\quad$ then $\boxed{f(0,t) = Ke^{-r(T-t)}}$

$\boxed{\text{European call}}$

at $S=0$ $\quad \frac{\partial f}{\partial t} = rf$ $\quad$ but $f(0,T) = 0$ $\quad \boxed{f(0,t) = 0}$

at $S = S_{max}$ $\quad f(S_{max}, t) = S_{max} - k e^{-(T-t)r}$