Note Title 4/6/2015

Crank-Nicholson (to solve $u_t = u_{xx}$) k = time step h = space step

$$\frac{U_i^{n+1} - U_i^n}{k} = \frac{1}{2}\left(\frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{h^2}\right) + \frac{1}{2}\left(\frac{U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}}{h^2}\right)$$

Let $r = \frac{k}{2h^2}$, then

$$-rU_{i-1}^{n+1} + (1+2r)U_i^{n+1} - rU_{i+1}^{n+1} = rU_{i-1}^n + (1-2r)U_i^n + rU_{i+1}^n \qquad 1 \le i \le M-1$$

$U_0^{n+1} = g_0(t_{n+1}) \qquad U_M^{n+1} = g_1(t_{n+1}) \qquad g_1$ and g_1 are given

This is an implicit method

Matrix form

$$\underbrace{\begin{bmatrix} (1+2r) & -r & 0 & \cdots & \cdots & 0 \\ -r & (1+2r) & -r & & & \vdots \\ 0 & -r & (1+2r) & -r & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & -r \\ 0 & \cdots & \cdots & 0 & -r & (1+2r) \end{bmatrix}}_{\text{positive definite symmetric matrix, banded with bandwidth 1}} \begin{bmatrix} U_1^{n+1} \\ \vdots \\ \vdots \\ U_{M-1}^{n+1} \end{bmatrix} = \begin{bmatrix} r\left[g_0(t_n) + g_0(t_{n+1})\right] + (1-2r)U_1^n + rU_2^n \\ \vdots \\ rU_{i-1}^n + (1-2r)U_i^n + rU_{i+1}^n \\ \vdots \\ rU_{M-2}^n + (1-2r)U_{M-1}^n + r(g_1(t_n) + g_1(t_{n+1})) \end{bmatrix}$$

use Cholesky

<u>Local truncation errors</u>

<u>Def</u>: the local truncation error is obtained by inserting the exact solution $u(x,t)$ of the PDE into the scheme and determine

by how much it fails to satisfy the discrete equations.

Method 1

$$\tau(x,t) = \frac{u(x,t+k)-u(x,t)}{k} - \frac{1}{h^2}\left[u(x-h,t)-2u(x,t)+u(x+h,t)\right]$$

truncation error

$$u(x,t+k) \approx u(x,t) + u_t(x,t)\,k + \frac{1}{2}u_{tt}(x,t)\,k^2$$

$$u(x-h,t) \approx u(x,t) - u_x(x,t)\,h + \frac{1}{2}u_{xx}(x,t)\,h^2 - \frac{1}{3!}u_{xxx}(x,t)\,h^3 + \frac{1}{4!}u_{xxxx}(x,t)\,h^4$$

$$u(x+h,t) \approx u(x,t) + u_x(x,t)\,h + \frac{1}{2}u_{xx}(x,t)\,h^2 + \frac{1}{3!}u_{xxx}(x,t)\,h^3 + \frac{1}{4!}u_{xxxx}(x,t)\,h^4$$

$$\tau(x,t) \approx \left(u_t(x,t) + \frac{1}{2}u_{tt}(x,t)\,k\right) - \left(u_{xx}(x,t) + \frac{1}{12}u_{xxxx}(x,t)\,h^2\right)$$

Since u is a solution of the PDE, we have $u_t(x,t) = u_{xx}(x,t)$

then $\boxed{u_{tt}(x,t) = u_{xxt}(x,t) = u_{xxxx}(x,t)}$

$$\tau(x,t) \simeq u_{xxxx}(x,t)\left(\frac{k}{2} - \frac{h^2}{12}\right)$$

this method is of order $O(h^2+k)$

Truncation error for Crank-Nicholson

$$\tau(x,t) = \frac{u(x,t+k) - u(x,t)}{k} - \frac{1}{2h^2}\left[u(x-h,t) - 2u(x,t) + u(x+h,t)\right] - \frac{1}{2h^2}\left[u(x-h,t+k)\right.$$

$$\left. - 2u(x,t+k) + u(x+h,t+k)\right]$$

1 $$\frac{u(x,t+k)-u(x,t)}{k} \approx u_t(x,t) + \frac{k}{2}u_{tt}(x,t) + \frac{k^2}{3!}u_{ttt}(x,t)$$

$-\frac{1}{2}$ $$\frac{u(x-h,t)-2u(x,t)+u(x+h,t)}{h^2} \approx u_{xx}(x,t) + \frac{1}{12}u_{xxxx}(x,t)\,h^2$$

$-\frac{1}{2}$ $$\frac{u(x-h,t+k)-2u(x,t+k)+u(x+h,t+k)}{h^2} \approx u_{xx}(x,t+k) + \frac{h^2}{12}u_{xxxx}(x,t+k)$$

$$\approx u_{xx}(x,t) + u_{xxt}(x,t)\,k + \frac{1}{2}u_{xxtt}(x,t)\,k^2 + \frac{1}{12}u_{xxxx}(x,t)\,h^2$$

$$\tau(x,t) \approx \left(u_t(x,t) + \frac{k}{2}u_{tt}(x,t) + \frac{k^2}{6}u_{ttt}(x,t)\right) - \frac{1}{2}\left(2u_{xx}(x,t) + \frac{h^2}{6}u_{xxxx}(x,t)\right.$$

$$\left. + u_{xxt}(x,t)\,k + \frac{1}{2}u_{xxtt}(x,t)\,k^2\right)$$

Use $u_t(x,t) = u_{xx}(x,t)$ $\quad u_{tt}(x,t) = u_{xxxx}(x,t)$

$u_{ttt}(x,t) = u_{xxxxxx}(x,t)$ $\quad u_{xxt}(x,t) = u_{xxxx}(x,t)$

$u_{xxtt}(x,t) = u_{xxxxxx}(x,t)$

$$\tau(x,t) \simeq \left(u_{xx} + \frac{k}{2} u_{xxxx} + \frac{k^2}{6} u_{xxxxxx}\right) - \frac{1}{2}\left(2u_{xx} + \frac{h^2}{6} u_{xxxx} + \right.$$

$$\left. + k\, u_{xxxx} + \frac{1}{2} u_{xxxxxx}\, k^2\right) = u_{xxxx}\left(-\frac{h^2}{12}\right) + u_{xxxxxx}\left(-\frac{k^2}{12}\right)$$

$$= O(h^2 + k^2)$$

<u>Von</u> <u>Neuman</u> <u>analysis</u>

Method 1

$$U_i^{n+1} = U_i^n + \frac{k}{h^2}\left(U_{i-1}^n - 2U_i^n + U_{i+1}^n\right)$$

$$U_0^{n+1} = g_0(t_{n+1}) \qquad U_M^{n+1} = g_1(t_{n+1})$$

$$U_i^0 = \eta(x_i)$$

Assume that because of machine precision or other sources of error, we have an error in the initial condition, i.e. we solve

$$\tilde{U}_i^{n+1} = \tilde{U}_i^n + \frac{k}{h^2}\left(\tilde{U}_{i-1}^n - 2\tilde{U}_i^n + \tilde{U}_{i+1}^n\right)$$

$$\tilde{U}_0^{n+1} = g_0(t_{n+1}) \qquad \tilde{U}_M^{n+1} = g_1(t_{n+1})$$

$$\tilde{U}_i^0 = \eta(x_i) + \delta(x_i) \qquad \text{with } \delta(x_i) \text{ small}$$

Assume all the rest of the calculations are exact.

$$\underbrace{\tilde{U}_i^n}_{\substack{\text{we}\\ \text{compute}}} = \underbrace{U_i^n}_{\substack{\text{we}\\ \text{want}}} + \underbrace{z_i^n}_{\text{error}}$$

z_i^n satisfies

$$z_i^{n+1} = z_i^n + \frac{k}{h^2}\left(z_{i-1}^n - 2z_i^n + z_i^n\right)$$

$$z_0^{n+1} = z_M^{n+1} = 0$$

$$z_i^0 = \delta(x_i)$$

Fact 1: $\begin{bmatrix} z_1^0 \\ \vdots \\ z_{M-1}^0 \end{bmatrix} \in \mathbb{R}^{M-1}$

$$V_1 = \begin{bmatrix} \sin\left(\frac{\pi}{M}\right) \\ \sin\left(\frac{\pi}{M}2\right) \\ \vdots \\ \sin\left(\frac{\pi}{M}(M-1)\right) \end{bmatrix} \quad V_2 = \begin{bmatrix} \sin\left(2\frac{\pi}{M}\right) \\ \sin\left(\frac{2\pi}{M}2\right) \\ \vdots \\ \sin\left(\frac{2\pi}{M}(M-1)\right) \end{bmatrix} \quad \cdots\cdots \quad V_j = \begin{bmatrix} \sin\left(\frac{j\pi}{M}\right) \\ \sin\left(\frac{j\pi}{M}2\right) \\ \\ \sin\left(\frac{j\pi}{M}(M-1)\right) \end{bmatrix}$$

$V_1, V_2, \ldots, V_{M-1}$ are $M-1$ vectors in $\mathbb{R}^{M-1}$ THEY ARE LINEARLY INDEPENDENT