Note Title 4/22/2015

Errors in monte carlo.

If we want an error less than $\varepsilon$ with probability $1-\delta$:

1) Find $z$ such that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-t^2/2}\,dt = 1-\frac{\delta}{2}$$

2) $n$, the number of samples, should satisfy

$$n \geq \frac{z^2\sigma^2}{\varepsilon^2}$$

Obs: To get $\sigma^2$, first take $n = n_1$, for some $n_1$ chosen, like $n_1 = 100$. Then compute

$$\bar{\mu} = \frac{\sum_{i=1}^{n_1} g(x_i)}{n_1}$$

$$\sigma^2 \approx \bar{\sigma}^2 = \frac{1}{n_1} \sum_{i=1}^{n_1} \left(g(x_i) - \bar{\mu}\right)^2$$ ($\sigma$ is not the volatity)

If $n_1 \geq \frac{z^2 \bar{\sigma}^2}{\varepsilon^2}$, then we are done.

Otherwise, let $n_2 = \frac{z^2 \bar{\sigma}^2}{\varepsilon^2}$

using Monte Carlo

Algorithm (to compute European option value at time $t=0$)

Input: $S_0 = S(0)$, $r$, $\sigma$, $T$, $K$, $\varepsilon$, $p$, type

We want an error less than $\varepsilon$ with probability $p$

type = 1 if call or 0 if put

$\delta = 1 - p$

$n_{min} = 100$ (minimum number of samples)

Compute $z$ such that $\int_{-\infty}^{z} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt = 1 - \frac{\delta}{2}$

$n = 0$ ($n$ = # of samples so far)

$i = 0$

while ($i = 0$)

while ($n < n_{min}$)

$n = n+1$

pick $x$ with $\mathcal{N}(0,1)$

$S_n = S_0 \exp((r - \sigma^2/2)T + \sigma\sqrt{T}x)$

end

$$f = \frac{\sum_{j=1}^{n} G(S_j, type)}{n}$$

$$\alpha = \frac{\sum_{j=1}^{n} (G(S_j, type) - f)^2}{n}$$

$$G(S, type) = \begin{cases} \max\{S-K, 0\} & \text{if } type = 1 \\ \max\{K-S, 0\} & \text{if call} \end{cases}$$

$$n_{min} = \left(\frac{z^2 \alpha}{\varepsilon^2}\right) 1.1$$

if $(n > n_{min})$ then

$i = 1$

endif

end

$f = f \exp(-rT)$ output

Geometric Brownian motion

$$dS = \mu S\,dt + \sigma S\,dW$$

W is Wiener process

Given $S(0) = S_0$ we have

$$S(T) = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{T}\,X\right)$$

with $X \sim N(0,1)$

But: 1) What if we want to solve a different stochastic differential equation that we can not solve explicity or

2) What if we need $S(t)$ for $0 \le t \le T$ for one realization.

Solving Ito stochastic differential equations numerically

$$dS = a(S,t)\,dt + b(S,t)\,dW$$

$W$ Wiener

Euler scheme

Select $\Delta t$ time step

$t_n = n\Delta t$

$S_n \simeq S(t_n)$ (following a realization path)

$x_n$ = randomly selected number with $N(0,1)$

Discretization $dW = W(t+dt) - W(t) \sim N(0, dt)$

$$\boxed{S_{n+1} - S_n = a(S_n, t_n)\,\Delta t + b(S_n, t_n)\sqrt{\Delta t}\,x_n}$$

Discretization applied to GBM

$$dS = \mu S\, dt + \sigma S\, dW$$

$$S_{n+1} = (1+\mu\Delta t) S_n + \sigma S_n \sqrt{\Delta t}\; x_n \quad \swarrow N(0,1)$$

This means

$$S_{n+1} \sim N\left((1+\mu\Delta t) S_n,\ \sigma^2 S_n^2 \Delta t\right)$$

$S_{n+1}$ is normal but it is suppose to be lognormal

Trick to avoid this. write GBM as

$$d \log S = \left(\mu - \tfrac{1}{2}\sigma^2\right) dt + \sigma\, dW$$

and now discretize

$$\log S_{n+1} - \log S_n = \left(\mu - \tfrac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\; x_n$$

thus

$$S_{n+1} = S_n \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\; x_n\right)$$

Notation: A realization of the asset price $S(t)$ for $0 \le t \le T$ is called an asset path price.

Example: Pricing of Asian options

In Asian options we have a set of prediterminned times $0 \le T_1 < T_2 < T_3 < \dots < T_n \le T$. Let $\bar{S} = \frac{1}{n} \sum_{i=1}^{n} S(T_i)$

$$\text{Pay off} = \begin{cases} \max\{\bar{S}-K, 0\} & \text{if call} \\ \max\{K-\bar{S}, 0\} & \text{if put} \end{cases}$$

algorithm (to compute $\bar{S}$)

Input: $S = S(0)$, $n$, $T_1, T_2, \ldots, T_n$ and $\Delta t$ Output $\bar{S}$ in $A$

$A = 0$

$t = 0$

for $i = 1:n$

while $(t + \Delta t < T_i)$

pick $x$ with $N(0,1)$

$$S = S\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\, x\right)$$

$t = t + \Delta t$

end

$\alpha = T_i - t$

pick $x$ with $\mathcal{N}(0,1)$

$$S = S\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\alpha + \sigma\sqrt{\alpha}\, x\right)$$

$t = T_i$

$A = A + S$

end

$$A = \frac{A}{n}$$

To compute the value of Asian options, use the algorithm for Europeans, but instead of using $S_n$, use $\bar{S}$ ($A$ in the above algorithm) but use $r$ instead of $\mu$ in the algorithm above.