Reminder from last class

~~Monte Carlo method to compute the~~

value of European options.

~~Value of option at expiry T is~~

$\cancel{f_T(S) =}$ K = strike price

S = value of the asset

$f_T(S)$ = value of the option at expiration time T if the value of the asset is S at time T.

$$f_T(S) = \begin{cases} \max\{0, S-K\} & \text{if call option} \\ \max\{0, K-S\} & \text{if put option} \end{cases}$$

$f(S)$ = value of option at time $t=0$ if the value of the asset at time $t=0$ is S.

Then

$$f(S) = e^{-rT}\,\hat{E}[f_T(S)]$$

$\hat{E}$ is the expectation with respect to the risk free probability.

If S follows a geometric Brownian motion, replace drift μ by risk-free rate r to get

$$S(T) = S(0)\, e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,X},$$

where $X \sim \mathcal{N}(0,1)$

Monte carlo to compute $f(S)$

1) Compute $x_1, x_2, \ldots$ random numbers ~~two~~ from a standard normal distribution $\mathcal{N}(0,1)$

2) ~~$\hat{E}[f_T] \approx$~~ $S_i = S(0)\, e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,x_i}$

3) $\hat{E}[f_T] \approx E = \frac{1}{N}\left[\sum_{i=1}^{N} f_T(S_i)\right]$

4) ~~$f(S)$~~ $e^{-rT} E$

Generating random numbers with standard normal distribution if the software being ~~Option 1:~~ used can only generate random numbers with uniform distribution in $(0,1)$

Obs: 1) X is a random variable with standard normal distribution $X \sim N(0,1)$ if

$$P(X \le x) = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt \ \text{ for all } x \in \mathbb{R}$$

$\stackrel{\wedge}{=} F(x)$

2) $Y \sim U(0,1)$ if $P(Y \le y) = y$ for all $0 \le y \le 1$

Obs: Assume the relation

$$Y = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X} e^{-t^2/2}\, dt = F(X),$$

where $F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$. Then

$Y \sim U(0,1) \Leftrightarrow X \sim N(0,1)$

Pf: Let $0 \le y \le 1$ ~~then~~ let x be the solution of $y = F(x)$, then

$Y \le y \iff X \le x$

then $P(Y \le y) = P(X \le x)$

Thus, if $Y \sim U(0,1) \iff$

$\forall y \in [0,1]$, $y = P(Y \le y) = P(X \le x) = F(x)$

$\iff X \sim N(0,1)$

Algorithm (to generate a number randomly with standard normal distribution)

Output ~~a random #~~ # in x

~~Pick y with $U(0,1)$ (to~~

Pick y randomly with uniform prob dist in $(0,1)$

Solve $y = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt$ for x.

~~Algorit~~ Solving $y = F(x) = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt$

Let $G(x) = F(x) - y$. y is fixed. Find the root of G using NR or bisection

Obs: $G(x) = \int_0^x \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt + \frac{1}{2} - y$

$$G'(x) = \frac{\cancel{e^{-x^2/2}}\, e^{-x^2/2}}{\sqrt{2\pi}}$$

~~Computing~~ $G(x)$ using any method to compute integrals, like trapezoid, Note, if $x<0$

$$\int_0^x \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt = -\int_x^0 \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt$$

Reminder of monte carlo S, random variable. $f:\mathbb{R}\to\mathbb{R}$. Generate $S_1, S_2, \ldots$ numbers randomly with the distribution of S, then

1 $$\boxed{E[f(S)] \approx \frac{\sum_{i=1}^{N} f(S_i)}{N}}$$

<u>Relation to integrals</u>; if $S \sim U(0,1)$,

then ~~and~~ $f: \mathbb{R} \to \mathbb{R}$, then

(2) $E[g(S)] = \int_0^1 g(s)\,ds$,

thus, if $S_1, S_2, \dots$ are randomly generated numbers with uniform prob distribution in $(0,1)$, then $\int_0^1 g(s)\,ds \approx \frac{1}{N}\left[\sum_{i=1}^{N} g(S_i)\right]$

<u>Question</u>: How large ~~sho~~ should be N in (1) or (2)?

<u>Probability</u>: Central limit theorem

$X_1, X_2, \dots$ i.i.d. $\quad g: \mathbb{R} \to \mathbb{R}$

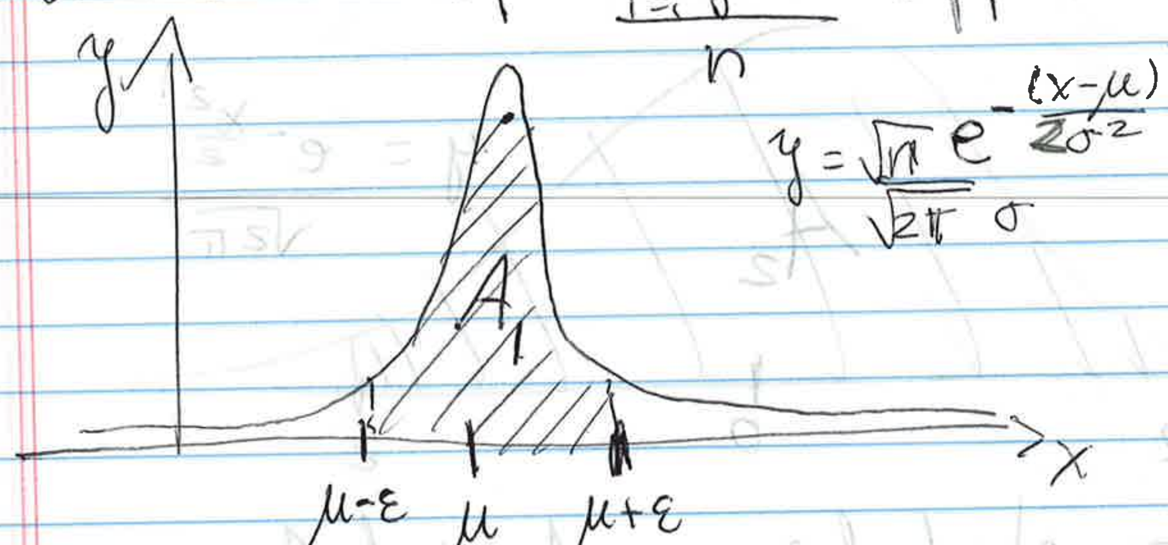
$\mu = E[g(X_i)] \quad \sigma^2 = Var[g(X_i)] = E[(g(X_i) - \mu)^2]$,

then, as $n \to \infty$

$$\frac{\sum_{i=1}^{n} g(X_i)}{n} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

<u>Goal</u>: Find μ, with an error at most ε with probability $1-\delta$,

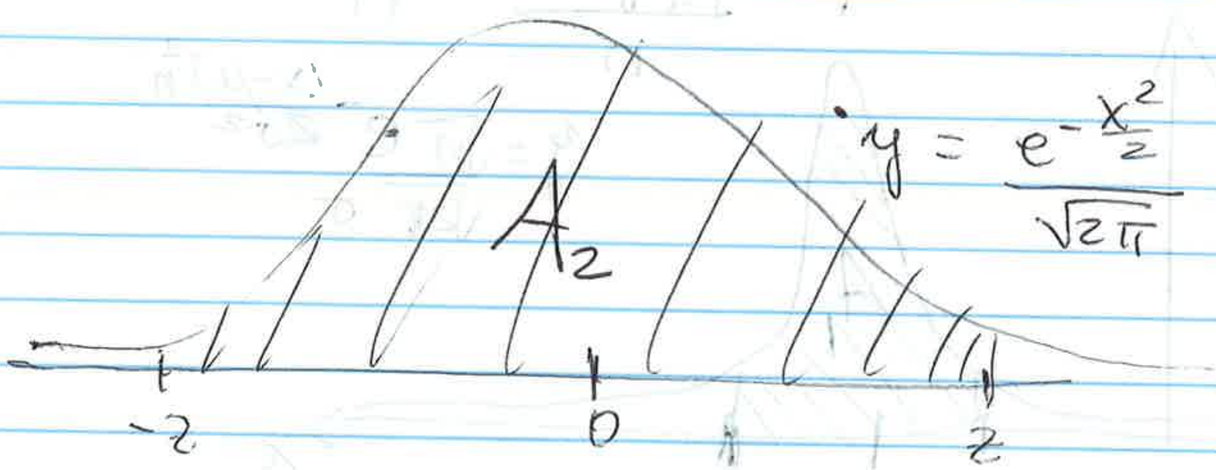
distribution of $\frac{\sum_{i=1}^{n} g(X_i)}{n}$ approx



the larger n, the bigger the shaded area. How large should n be so that the area is $1-\delta$? that would mean that if we sample $\bar{\mu} = \frac{\sum_{i=1}^{n} g(x_i)}{n}$ for such n, with probability $1-\delta$, $|\bar{\mu}-\mu| < \varepsilon$.

x_i sample of X_i

Distribution of $X \sim N(0,1)$, i.e. distribution of a ~~random variable~~ normal standard random variable



The two shaded areas A_1 and A_2 are equal

, $A_1 = A_2 \iff \varepsilon = z\sqrt{\frac{\sigma^2}{n}}$

So given ~~ε~~ δ, first get z such that $A_2 = 1-\delta$, then given ε set n such that

$$n \geq \frac{z^2\sigma^2}{\varepsilon^2}$$

So we need to get z and σ^2.