Note Title 4/15/2015

Obs: need to update b every time step, but A stays the same

Obs: A can be stored in an $M-1 \times 3$ matrix $\tilde{A}$

$\tilde{a}_{i1} = a_{i\,i-1}$ $\quad$ $\tilde{a}_{i2} = a_{ii}$ $\quad$ $\tilde{a}_{i3} = a_{i\,i+1}$

$2 \le i \le M-1$ $\quad$ $1 \le i \le M-1$ $\quad$ $1 \le i \le M-2$

American options Let $g_i = \begin{cases} K - i\Delta S & \text{if a put} \\ i\Delta S - K & \text{if a call} \end{cases}$

We have f_i^n $0 \le i \le M$ and use the following to find f_i^{n-1} $0 \le i \le M$

$$x_i^{(0)} = f_i^n$$

One iteration convention $a_{10} = 0$ $a_{M-1\,M} = 0$

for $i = 1 : M-1$

$$x_i^{(k+1)} = \max\left\{ g_i , \frac{b_i - a_{i\,i-1} x_{i-1}^{(k+1)} - a_{i\,i+1} x_{i+1}^{(k)}}{a_{ii}} \right\}$$

end

do as many iterations as necessary so that $\| x^{(k)} - x^{(k+1)} \| < \varepsilon$

for some small ε prescribed

Probability $\boxed{\text{Strong law of large numbers}}$

$X_i : \Omega \to \mathbb{R}$ $\quad X_1, X_2, \dots$ random variables i.i.d.
(independent identically distributed). Assume $E[X_i] = \mu$
Then, with probability 1,

$$\lim_{N \to \infty} \frac{\sum_{i=1}^{N} X_i}{N} = \mu$$

Obs: X random variable. X uniformly distributed in $(0,1)$. $X \sim U(0,1)$ means $P(a \le X \le b) = b-a$ for all $0 \le a \le b \le 1$. If $g : (0,1) \to \mathbb{R}$ then

(1) $E[g(X)] = \int_0^1 g(x)\,dx$

<u>Obs</u>: The strong law of large numbers plus eq (1) imply:
If we pick numbers $x_1, x_2, \ldots$ at random with uniform probability distribution in $(0,1)$, then

$$(2) \quad \lim_{N\to\infty} \frac{\sum_{i=1}^{N} g(x_i)}{N} = \int_0^1 g(x)\,dx$$

<u>Monte Carlo method</u>: To compute $\int_0^1 g(x)\,dx$, pick $x_1, x_2, \ldots$ randomly with uniform distribution in $(0,1)$ and set

$$\int_0^1 g(x)\,dx \approx \frac{\sum_{i=1}^{N} g(x_i)}{N}$$

Obs: 1) For 1-d integrals (1 variable), deterministic methods like trapezoid, etc work better than Monte Carlo methods
2) In high dimensions, deterministic methods become computationally expensive and Monte Carlo are a good alternative.
3) $I = \int_a^b \phi(x)\,dx$. Pick $x_1, x_2, \ldots$ randomly, uniformly, but in (a,b) and set $I \approx \left(\frac{1}{N}\sum_{i=1}^{N} \phi(x_i)\right)(b-a)$

4) If $x_1, x_2, \ldots$ are randomly picked numbers in $(0,1)$ with uniform distribution, then $y_i = a + (b-a)x_i$ gives $y_1, y_2, \ldots$ randomly picked numbers in (a,b)

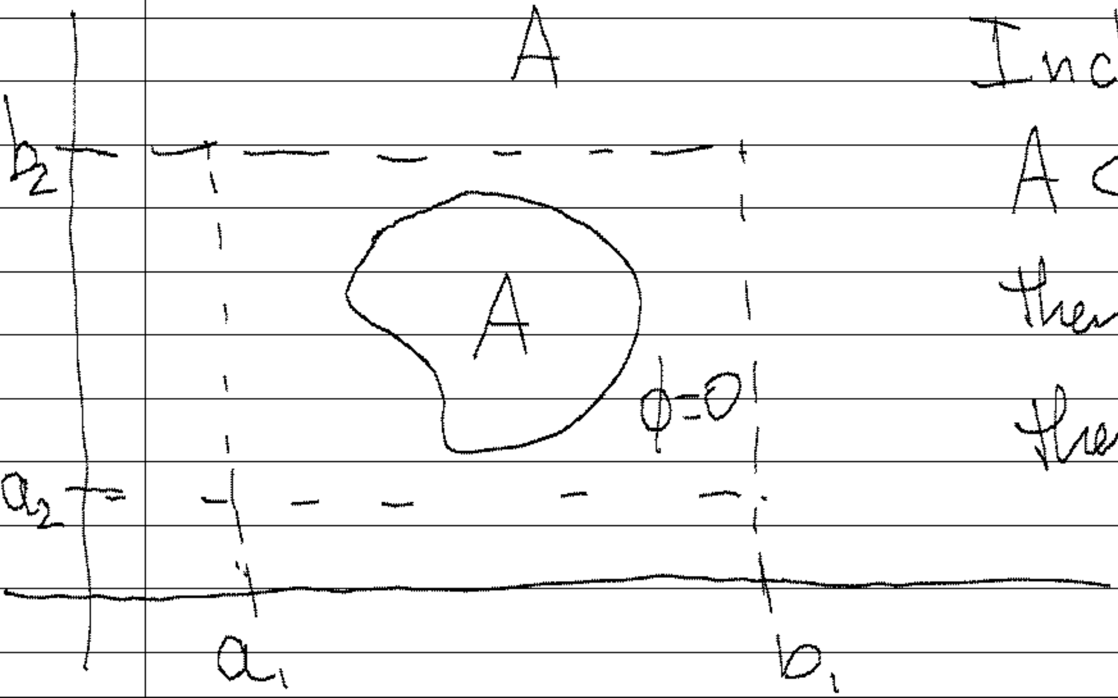
5) $I = \int_A \phi(x)\,dx \qquad A \subseteq \mathbb{R}^n$

Include A in a rectangle (assume $n=2$)

$A \subset [a_1, b_1] \times [a_2, b_2]$

then select $x_1, x_2, \ldots$ rand unif in (a_1, b_1)

then select $y_1, y_2, \ldots$ rand unif in (a_2, b_2)



$$I \approx \left(\frac{\displaystyle\sum_{\substack{1 \le i \le N \\ (x_i, y_i) \in A}} \phi(x_i, y_i)}{N} \right) (b_1 - a_1)(b_2 - a_2)$$

<u>Example</u>: Price of an European option at $t=0$ is:

$$\boxed{f = e^{-rT} \, \hat{E}\left[f_T \right]}$$

T = maturity time

$\boxed{f_T}$ is the payoff at maturity $= \boxed{\begin{cases} \max\{0, S(T) - K\} & \text{if call} \\ \max\{0, K - S(T)\} & \text{if put} \end{cases}}$

f value of the option at time $t=0$

$\hat{E}[\ ]$ = expected value with respect to the risk free rate

r probability

If the asset price follows Geometric Brownian motion (replace drift μ by risk-free interest)

$$dS = rS\,dt + \sigma S\,dW \qquad W \text{ Weiner}$$

then

$$S(T) = S(0)\, e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\phi}$$

where $\phi \sim N(0,1)$, i.e. ϕ is a random variable with standard normal distribution, i.e.

$$P(a \le \phi \le b) = \int_a^b \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt$$

Let $x_1, x_2, \ldots$ be numbers picked randomly with standard normal distribution. Then set $S_i = S(0)\, e^{(r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}\, x_i}$,

then

$$\boxed{\hat{E}[f_T] \approx \frac{1}{N}\left(\sum_{i=1}^{N} f_T(S_i)\right)}$$

where

$$\boxed{f_T(S) = \begin{cases} \max\{0, S-K\} & \text{call} \\ \max\{0, K-S\} & \text{put} \end{cases}}$$

then $\boxed{f \approx e^{-rT}\, \hat{E}[f_T]}$
