Note Title 4/13/2015

Boundary conditions Discretization

European Put

$S_{max} = M\Delta S$

$f_i^n \simeq f(i\Delta S, n\Delta t)$ $f = f(S,t)$ $T = N\Delta t$

Terminal condition $f_i^N = \max\{K - i\Delta S, 0\}$ $0 \le i \le M$

Boundary condition at $S=0$ $f_0^n = K e^{-r(N-n)\Delta t}$ $0 \le n \le N$

Boundary condition at $S = S_{max}$ $f_M^n = 0$

European Call

Terminal condition $\boxed{f_i^N = \max\{ i\Delta S - K, 0\}}$ $0 \le i \le M$

Boundary condition at $S=0$ $f_0^n = 0$

Boundary condition at $S=S_{max}$ $f_M^n = M\Delta S - K e^{-r(N-n)\Delta t}$ $0 \le n \le N$

Method I. Explicit

We have to solve backward in time, we use backward difference for $\frac{\partial f}{\partial t}$

$$\frac{\partial f}{\partial t}(i\Delta S, n\Delta t) \approx \frac{f_i^n - f_i^{n-1}}{\Delta t}$$

Central difference for $\frac{\partial f}{\partial S}$

$$\frac{\partial f}{\partial S}(i\Delta S, n\Delta t) \approx \frac{f_{i+1}^n - f_{i-1}^n}{2\Delta S}$$

$$\frac{\partial^2 f}{\partial S^2}(i\Delta S, n\Delta t) \approx \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{(\Delta S)^2}$$

Plug these expressions into Black Scholes to get

$$\frac{f_i^n - f_i^{n-1}}{\Delta t} + r i \cancel{\Delta S} \frac{\left(f_{i+1}^n - f_{i-1}^n\right)}{2 \cancel{\Delta S}} + \frac{1}{2} \sigma^2 i^2 \cancel{(\Delta S)^2} \frac{\left(f_{i+1}^n - 2 f_i^n + f_{i-1}^n\right)}{\cancel{(\Delta S)^2}} = r f_i^n$$

$$\boxed{f_i^{n-1} = \left(1 - r\Delta t - \sigma^2 i^2 \Delta t\right) f_i^n + \frac{r i \Delta t}{2}\left(f_{i+1}^n - f_{i-1}^n\right) + \frac{\sigma^2 i^2 \Delta t}{2}\left(f_{i+1}^n + f_{i-1}^n\right)}$$

Error $O\left(\Delta t + (\Delta S)^2\right)$

Explicit method. For stability, we need

$$\Delta t \le \frac{(\Delta S)^2}{2} \frac{1}{\frac{1}{2}\sigma^2 S_{max}^2} \qquad \boxed{\Delta t \le \frac{(\Delta S)^2}{\sigma^2 S_{max}^2}}$$ too restrictive

<u>Implicit method</u>

$$\frac{\partial f}{\partial t}(i\Delta S, n\Delta t) \approx \frac{f_i^{n+1} - f_i^n}{\Delta t}$$ forward difference

$$\frac{\partial f}{\partial S}(i\Delta S, n\Delta t) \approx \frac{f_{i+1}^n - f_{i-1}^n}{2\Delta S}$$ central difference

$$\frac{\partial^2 f}{\partial S^2}(i\Delta S, n\Delta t) \approx \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{(\Delta S)^2}$$

Plug into Black-Scholes equation to get

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + r i \cancel{\Delta S}\, \frac{f_{i+1}^n - f_{i-1}^n}{2\cancel{\Delta S}} + \frac{\sigma^2}{2} i^2 \cancel{(\Delta S)^2}\, \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{\cancel{(\Delta S)^2}} = r f_i^n$$

f_i^{n+1} is known, but f_{i-1}^n, f_i^n and f_{i+1}^n are not known

$$\boxed{\begin{aligned}&\left(\frac{ri}{2}\Delta t - \frac{\sigma^2 i^2}{2}\Delta t\right) f_{i-1}^{n} + \left(1 + \sigma^2 i^2 \Delta t + r\Delta t\right) f_i^{n} \\ &+ \left(-\frac{ri}{2}\Delta t - \frac{\sigma^2 i^2}{2}\Delta t\right) f_{i+1}^{n} = f_i^{n+1}\end{aligned}} \qquad 1 \le i \le M-1$$

$$\boxed{f_0^n \text{ and } f_M^n \text{ given}}$$

This is a tridiagonal system that can be solved in $O(M)$ operations. This method is stable. $O(\Delta t + (\Delta S)^2)$

Crank-Nicolson: To keep stability of the implicit method just introduced, but to improve the error to $O((\Delta t)^2 + (\Delta S)^2)$

$$\frac{f_i^n - f_i^{n-1}}{\Delta t} + r i \Delta S \, \frac{1}{2}\left(\frac{f_{i+1}^{n-1} - f_{i-1}^{n-1}}{2\Delta S} + \frac{f_{i+1}^{n} - f_{i-1}^{n}}{2\Delta S} \right) + \frac{1}{2}\sigma^2 i^2 (\Delta S)^2$$

$$\frac{1}{2}\left(\frac{f_{i+1}^{n-1} - 2f_i^{n-1} + f_{i-1}^{n-1}}{(\Delta S)^2} + \frac{f_{i+1}^{n} - 2f_i^{n} + f_{i-1}^{n}}{(\Delta S)^2} \right) = r \, \frac{1}{2}\left(f_i^{n-1} + f_i^{n}\right)$$

Let $\alpha_i = \frac{\Delta t}{4}(\sigma^2 i^2 - r i)$, $\beta_i = -\frac{\Delta t}{2}(\sigma^2 i^2 + r)$ and $\gamma_i = \frac{\Delta t}{4}(\sigma^2 i^2 + i r)$

Crank-Nicolson reduces to

$$\boxed{-\alpha_i f_{i-1}^{n-1} + (1-\beta_i) f_i^{n-1} - \gamma_i f_{i+1}^{n-1} = \alpha_i f_{i-1}^{n} + (1+\beta_i) f_i^{n} + \gamma_i f_{i+1}^{n}}$$

Again this is a tridiagonal system

American options

American put: Early exercise choice implies

$f(S,t) \geq K - S$

American call: $f(S,t) \geq S - K$

Boundary conditions: Put

for European put $f(0,t) = K e^{-r(T-t)}$ and $f(S_{max}, t) = 0$

$\boxed{\text{American Put} : f(0,t) = K \text{ and } f(S_{max}, t) = 0}$

For a call

European call: $f(0,t) = 0$ and $f(S_{max}, t) = S_{max} - K e^{-r(T-t)}$

$$\boxed{\text{American call: } f(0,t) = 0 \quad \text{and} \quad f(S_{max}, t) = S_{max} - K e^{-r(T-t)}}$$

$\boxed{\text{Terminal conditions do not change}}$

Explicit method for American options

One time step discretized Black-Scholes

$$\tilde{f}_i^{n-1} = (1 - r\Delta t) f_i^n + \frac{r i \Delta t}{2} \left(f_{i+1}^n - f_{i-1}^n\right) + \frac{\sigma^2 i^2 \Delta t}{2} \left(f_{i-1}^n - 2 f_i^n + f_{i+1}^n\right)$$

$$f_i^{n-1} = \begin{cases} \max\{\tilde{f}_i^{n-1}, K - i\Delta S\} & \text{if a put option} \\ \max\{\tilde{f}_i^{n-1}, i\Delta S - K\} & \text{if a call option} \end{cases}$$
