Note Title 4/1/2015

Central differences: $\boxed{f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)}$

Check: $f(x+h) = f(x) + f'(x)h + \frac{1}{2} f''(x) h^2 + \frac{1}{3!} f'''(x) h^3 + O(h^4)$

$f(x-h) = f(x) - f'(x)h + \frac{1}{2} f''(x) h^2 - \frac{1}{3!} f'''(x) h^3 + O(h^4)$

Substract and devide by 2h

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{1}{6} f'''(x) h^2 + O(h^3)$$

Finite difference approximation of $f''(x)$

$$\boxed{f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)}$$

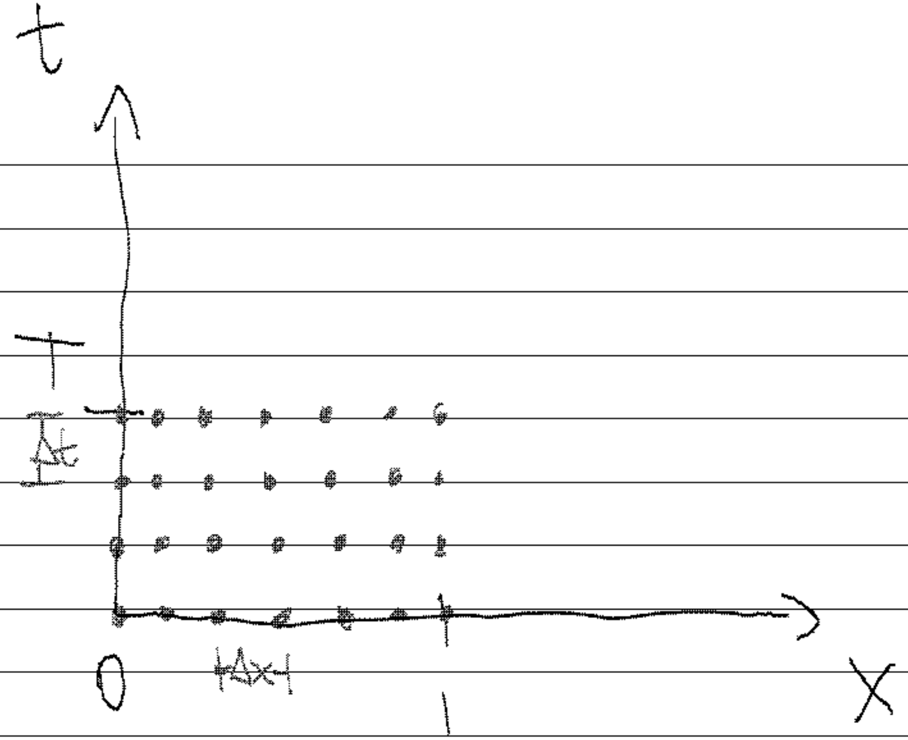
$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{3!}f'''(x)h^3 + \frac{1}{4!}f^{(iv)}(x)h^4 + O(h^5)$$

$$f(x-h) = f(x) - f'(x)h + \frac{1}{2}f''(x)h^2 - \frac{1}{3!}f'''(x)h^3 + \frac{1}{4!}f^{(iv)}(x)h^4 + O(h^5)$$

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + \frac{1}{12}f^{(iv)}(x)h^2 + \dots$$

Grid: Discretization of the region in the (x,t) plane (or (x,y)) where we seek to solve the PDE

Standard way: Fix Δx and Δt and set $(x_i, t_j) = (i\Delta x, j\Delta t)$ for i,j such that (x_i, t_j) is in the region where we want the solution



in $0 \le x \le 1$ and $T \ge t \ge 0$

Finite difference scheme

1) Replace the region in (x,t) plane where you want the solution by a grid

2) Replace derivatives of ϕ by finite difference approxima

tions, where ϕ is evaluated only at the grid points.

3) Do similar approximations for the boundary of the region of interest.

Let $u_{ij} \approx u(i\Delta x, j\Delta t)$. $u(x,t)$ is the exact solution to the PDE with the corresponding initial and boundary conditions

Steps 1), 2) & 3) lead to a large system of equations «algebraic» for the u_{ij}. The last step is to solve those equations

<u>Example</u> with an ode.

$y'' = -1$ $\quad y(0) = y(1) = 0$ $\quad$ we want $y(x)$ for $0 \le x \le 1$

<u>Grid</u>: $h = \frac{1}{N}$ $\quad x_i = ih$ $\quad 0 \le i \le N$

these are the grid points

$y_i \approx y(x_i)$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = -1 \qquad 1 \le i \le N-1$$

$$y_0 = y_N = 0$$

$N+1$ equations for $N+1$ unknowns, $y_0, y_1, \dots, y_N$

<u>Warning</u>: Not all finite difference schemes lead to the

exact solution by simply decreasing Δx and Δt

Example $$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad t > 0$$

$$u(x,0) = u_0(x) \qquad -\infty < x < \infty$$

The solution is $u(x,t) = u_0(x-t)$

Check $$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} u_0(x-t) = u_0'(x-t) \quad (-1)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} u_0(x-t) = u_0'(x-t)$$

Then $$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

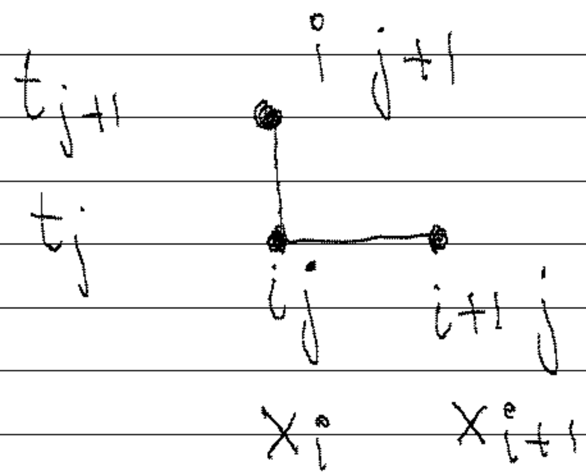
Example of a finite difference scheme that goes wrong

$x_i = ih \qquad t_j = jh \qquad u_{ij} \approx u(x_i, t_j)$

$$\frac{\partial u}{\partial t}(x_i, t_j) \approx \frac{u_{i\,j+1} - u_{ij}}{h} \qquad \frac{\partial u}{\partial x}(x_i, t_j) \approx \frac{u_{i+1\,j} - u_{ij}}{h}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \implies \frac{u_{i\,j+1} - u_{ij}}{h} + \frac{u_{i+1\,j} - u_{ij}}{h} = 0$$

$$u_{i\,j+1} = 2u_{ij} - u_{i+1\,j}$$



Example

Exact $u_0(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$

numerical approximation

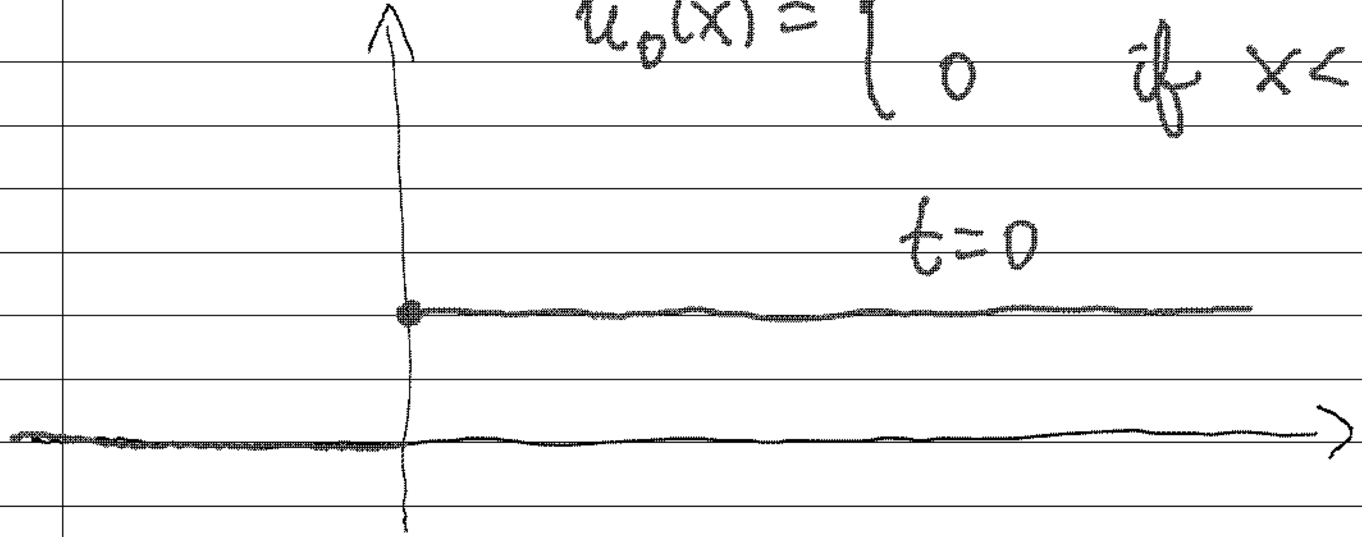
$i \geq 0$

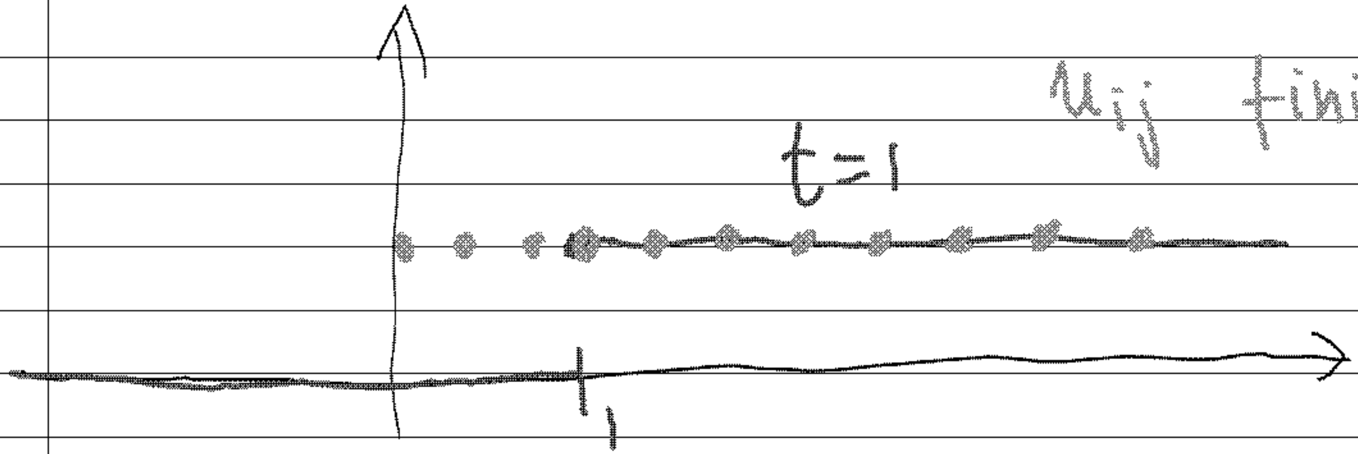
$u_{i1} = 2u_{i0} - u_{i+1\,0}$

$u_{i1} = 1$ for all $i \ge 0$

$u_{i2} = 1$ for all $i \ge 0$

$u_{ij} = 1$ for all $i \ge 0$ and all $j \ge 0$





u_{ij} finite difference approximation

Finite differences for the heat equation

$u_t = u_{xx} \qquad 0 \le x \le 1 \qquad t \ge 0$

$u(x,0) = \eta(x) \qquad 0 \le x \le 1$

$$\left.\begin{aligned} u(0,t) &= g_0(t) \\ u(1,t) &= g_1(t) \end{aligned}\right\} \quad t \ge 0$$

$x_i = ih \qquad t_n = nk \qquad h = \Delta x \qquad k = \Delta t \qquad h = \frac{1}{M} \qquad 0 \le i \le M$

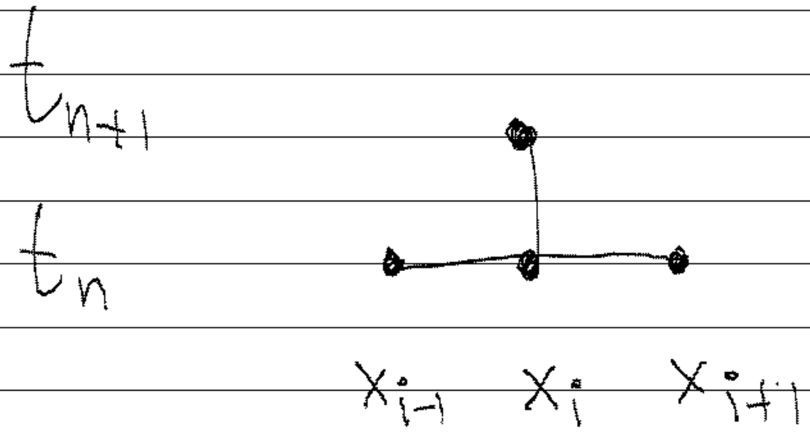
$n \ge 0$

$U_i^n \approx u(x_i, t_n)$

Method 1

$$\frac{U_i^{n+1} - U_i^n}{k} = \frac{1}{h^2}\left(U_{i-1}^n - 2U_i^n + U_{i+1}^n\right)$$

Stencil



We can write $U_i^{n+1} = U_i^n + \dfrac{k}{h^2}\left(U_{i-1}^n - 2U_i^n + U_{i+1}^n\right)$

This is an explicit method because U_i^{n+1} is given explicitly in terms the previous data, i.e. U_ℓ^n for some ℓs.

<u>Method 2</u> <u>Crank-Nicholson</u>

$$\frac{U_i^{n+1} - U_i^n}{k} = \frac{1}{2}\frac{\left(U_{i-1}^n - 2U_i^n + U_{i+1}^n\right)}{h^2} + \frac{1}{2}\frac{\left(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}\right)}{h^2}$$