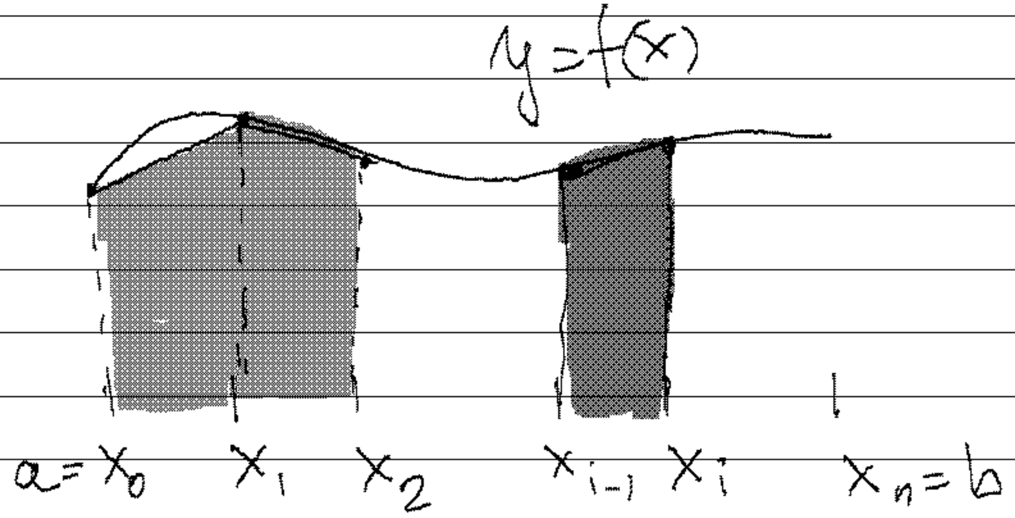
Note Title 3/9/2015

Numerical integration

Trapezoidal rule

$$x_i = a + \frac{i(b-a)}{n} \qquad 0 \le i \le n$$



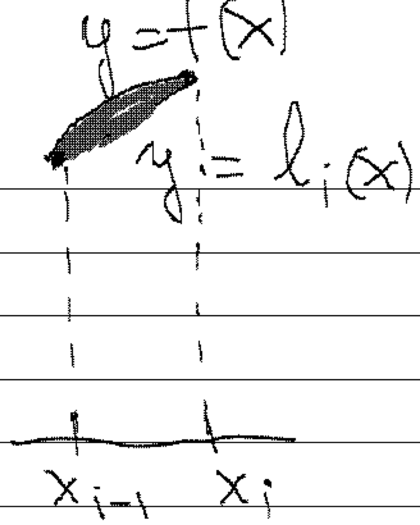
Area of Trapezoid =

$$= \frac{1}{2}(f_i + f_{i-1})(x_i - x_{i-1}),$$

where $f_i = f(x_i)$

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \frac{1}{2}(f_{i-1} + f_i)\frac{(b-a)}{n} = \frac{(b-a)}{n}\left(\frac{f_0}{2} + f_1 + \cdots + f_{n-1} + \frac{f_n}{2}\right)$$

Error in trapezoidal rule



In $[x_{i-1}, x_i]$

$$f(x) \approx f(x_{i-1}) + f'(x_{i-1})(x - x_{i-1}) + \frac{1}{2} f''(x_{i-1})(x - x_{i-1})^2 + \dots$$

$$l_i(x) = f(x_{i-1}) + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right)(x - x_{i-1})$$

Red area = error in $[x_{i-1}, x_i] = \int_{x_{i-1}}^{x_i} \left(f(x) - l_i(x) \right) dx$

$$l_i(x) \approx f(x_{i-1}) + \frac{\left[f(x_{i-1}) + f'(x_{i-1})(x_i - x_{i-1}) + \frac{f''}{2}(x_{i-1})(x_i - x_{i-1})^2 - f(x_{i-1}) \right]}{(x_i - x_{i-1})}(x - x_{i-1}) + \dots$$

$$l_i(x) \approx f(x_{i-1}) + f'(x_{i-1})(x - x_{i-1}) + \frac{1}{2} f''(x_{i-1})(x_i - x_{i-1})(x - x_{i-1}) + \cdots$$

$$f(x) - l_i(x) \approx \frac{1}{2} f''(x_{i-1}) \left[(x - x_{i-1})^2 - (x_i - x_{i-1})(x - x_{i-1}) \right] + \cdots$$

$$\boxed{\int_{x_{i-1}}^{x_i} \left(f(x) - l_i(x) \right) dx} \approx \frac{1}{2} f''(x_{i-1}) \left[\frac{(x - x_{i-1})^3}{3} - (x_i - x_{i-1}) \frac{(x - x_{i-1})^2}{2} \right]_{x_{i-1}}^{x_i}$$

$$\approx -\frac{1}{12} f''(x_{i-1}) (x_i - x_{i-1})^3 = \boxed{-\frac{1}{12} f''(x_{i-1}) \frac{(b-a)^3}{n^3}}$$

$$\text{Error} = \left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} \frac{(f_{i-1} + f_i)}{2} \frac{(b-a)}{n} \right| \approx O\left(\max_{a \le x \le b} |f''(x)| \right) \times O\left((b-a) h^2 \right)$$

where $h = x_i - x_{i-1} = \frac{b-a}{n}$

Richardson extrapolation

Assume we know a function $T(h)$ is of the form

$T(h) = a_0 + a_1 h^{p_1} + a_2 h^{p_2} + \cdots$

where $0 < p_1 < p_2 < \ldots$

We know $p_1, p_2, \ldots$

We can compute $T(h)$, but the smaller h, the more expensive is to compute $T(h)$.

Goal: Compute $a_0 = T(0)$

$T(h) = a_0 + a_1 h^{p_1} + a_2 h^{p_2} + \cdots$

$$T(h/2) = a_0 + a_1 \frac{h^{p_1}}{2^{p_1}} + a_2 \frac{h^{p_2}}{2^{p_2}} + \cdots$$

$$\frac{2^{p_1} T(h/2) - T(h)}{2^{p_1} - 1} = a_0 + \frac{\left(\frac{1}{2^{p_2 - p_1}} - 1\right)}{2^{p_1} - 1} a_2 h^{p_2} + \cdots$$

which is of the form $a_0 + a_2 O\left(\frac{1}{2^{p_1}} h^{p_2}\right)$, which is better that $T(h) = a_0 + a_1 O(h^{p_1})$

$$T_0(h) = T(h) = a_0 + a_1 h^{p_1} + a_2 h^{p_2} + a_3 h^{p_3} + \cdots$$

$$T_1(h) = \frac{2^{p_1} T_0(h/2) - T_0(h)}{2^{p_1} - 1} = a_0 + b_2 h^{p_2} + b_3 h^{p_3} + \cdots$$

$$T_2(h) = \frac{2^{p_2}\, T_1(h/2) - T_1(h)}{2^{p_2} - 1} = a_0 + c_3 h^{p_3} + \cdots$$

Table

$$\begin{array}{llll}
 & T_1(h) & & \\
 & \shortparallel & & \\
A_{00} = T_0(h) & & & \\
A_{01} = T_0(h/2) & A_{10} = \dfrac{2^{p_1} A_{01} - A_{00}}{2^{p_1} - 1} & & \\
A_{02} = T_0(h/4) & A_{11} = \dfrac{2^{p_1} A_{02} - A_{01}}{2^{p_1} - 1} & A_{20} = \dfrac{2^{p_2} A_{10} - A_{11}}{2^{p_2} - 1} & \\
A_{03} = T_0(h/8) & A_{12} = \dfrac{2^{p_1} A_{03} - A_{02}}{2^{p_1} - 1} & A_{21} = \dfrac{2^{p_2} A_{11} - A_{12}}{2^{p_2} - 1} &
\end{array}$$

Example $f(h) = \dfrac{1}{1+h^2}$

Note $f(h) = 1 - h^2 + h^4 - h^6 + \cdots$ $\quad p_1 = 2 \quad p_2 = 4 \quad p_3 = 6$

$h = 0.5$

$f(0.5) = 0.8$

$f\left(\frac{0.5}{2}\right) = 0.9412$ $\quad 0.988$

$f\left(\frac{0.5}{4}\right) = 0.9846$ $\quad 0.999\times\cdots$ $\quad 0.9997$

Algorithm (Richardson)

Input: $p_1, p_2, \ldots, p_k$ $\qquad A_{0j} = T(h/2^j)$

Output: A_{kk} = approximation of $T(0)$

for $i = 1:k$

for $j = 0:k-i$

$$A_{ij} = \frac{2^{p_i} A_{i-1\,j+1} - A_{i-1\,j}}{2^{p_i}-1}$$

end

end

Back to trapezoidal

$$I = \int_a^b f(x)\,dx \qquad h = \frac{b-a}{n} \qquad f_i = f(x_i)$$

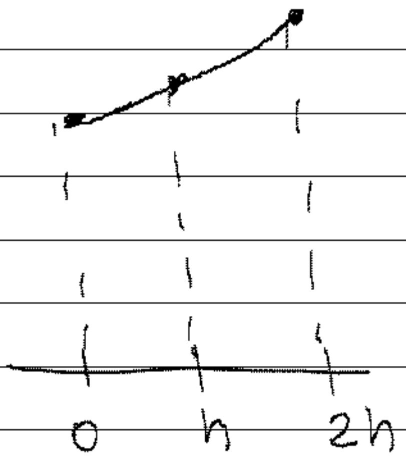
$$x_i = a + ih$$

$$T(h) = \left(\frac{1}{2} f_0 + f_1 + \cdots + f_{n-1} + \frac{f_n}{2}\right) h$$

$\underline{\text{Fact:}}$ $T(h) = I + \alpha_1 h^2 + \alpha_2 h^4 + \alpha_3 h^6 + \cdots$

So we can use Richardson extrapolation

<u>Simpson's rule</u> Interpolate the points $(0, f_0)$ (h, f_1) and $(2h, f_2)$ by a polynomial of degree 2.



$f_0 = f(0) \qquad f_1 = f(h) \qquad f_2 = (2h)$

$$P(x) = f_0 \frac{(x-h)(x-2h)}{(-h)(-2h)} + f_1 \frac{x(x-2h)}{h(-h)} + f_2 \frac{x(x-h)}{2h\,h}$$

$$\int_0^{2h} P(x)\,dx = 2h\left\{\frac{f_0}{6} + \frac{4}{6}f_1 + \frac{f_2}{6}\right\}$$

Simpson's rule $\quad n = 2k$

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\sum_{i=1}^{k}\left(f_{2i-2} + 4f_{2i-1} + f_{2i}\right)$$

$$h = \frac{b-a}{n}$$

$$\boxed{\text{Error} \approx O(h^4)}$$