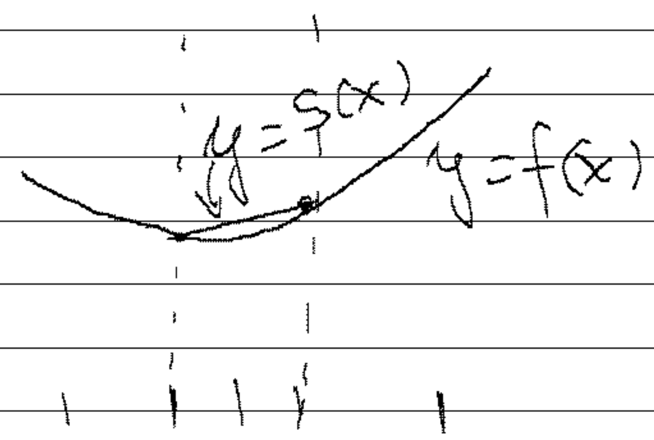
Splines: Error $d=1$

$x_i < \xi < x_{i+1}$



$a = x_0 \quad x_1 \quad x_i \quad \xi \quad x_{i+1} \quad x_n = b$

$x_{i+1} - x_i = h = \dfrac{b-a}{n}$

$S(x)$ = spline

$= 0$

$$f(\xi) - S(\xi) \approx f(x_i) - S(x_i) + $$
$$+ \left(f'(x_i) - S'(x_i)\right)(\xi - x_i) + \frac{1}{2} f''(x_i)(\xi - x_i)^2$$

$$S(\xi) = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(\xi - x_i)$$

$= S'(x_i)$

$S(x_i) = f(x_i)$

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{1}{2} f''(x_i)(x_{i+1} - x_i)^2$$

$$S'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \simeq f'(x_i) + \frac{1}{2} f''(x_i)(x_{i+1} - x_i)$$

$$f(\xi) - S(\xi) \simeq -\frac{1}{2} f''(x_i)(x_{i+1} - x_i)(\xi - x_i) + \frac{1}{2} f''(x_i)(\xi - x_i)^2$$

$$f(\xi) - S(\xi) \simeq \frac{1}{2} f''(x_i)(\xi - x_i)\left[\xi - x_i - x_{i+1} + x_i\right]$$

$$f(\xi) - S(\xi) \simeq \frac{-1}{2} f''(x_i) \underbrace{(\xi - x_i)(x_{i+1} - \xi)}_{\le \frac{h^2}{4}}$$

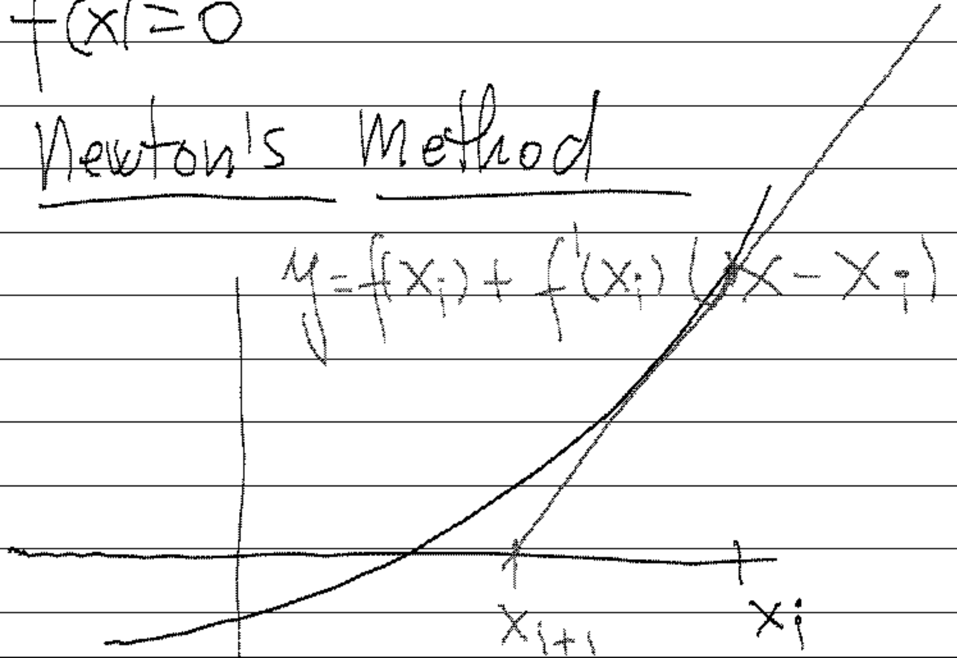
$$\text{Error} \simeq \max_{a \le \xi \le b} |f(\xi) - S(\xi)| \lesssim \left(\max_{a \le x \le b} |f''(x)|\right) \frac{h^2}{8}$$

$d=3$ cubic splines

$$\text{Error} \approx \max_{a \le \xi \le b} |f(\xi) - S(\xi)| \approx \left(\max_{a \le x \le b} |f^{(4)}(x)| \right) O(h^4)$$

Back to nonlinear equations, i.e. find the solution of $f(x)=0$

Newton's Method



Iterative method.

x_i is a guess of the zero

Replace $f(x)$ by its linear approximation at x_i

$$0 = f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

<u>Convergence</u>: Its convergence depends on the initial guess x_0 and the function $f(x)$. It will converge if you start close enough.

Let x^* be the root

$$\boxed{0 = f(x^*) \cong f(x_n) + f'(x_n)(x^* - x_n) + \frac{1}{2} f''(x_n)(x^* - x_n)^2}$$

$$x^* - x_{n+1} = x^* - x_n + \frac{f(x_n)}{f'(x_n)}$$

$$-\frac{1}{2} f''(x_n)(x^*-x_n)^2 \cong f(x_n) + f'(x_n)(x^*-x_n)$$

devide by $f'(x_n)$

$$-\frac{1}{2} \frac{f''(x_n)}{f'(x_n)} (x^*-x_n)^2 \simeq \frac{f(x_n)}{f'(x_n)} + (x^*-x_n) = x^*-x_{n+1}$$

$\varepsilon_n = |x^*-x_n|$ = error at step n

$$\varepsilon_{n+1} \simeq \frac{1}{2}\left|\frac{f''(x_n)}{f'(x_n)}\right| \varepsilon_n^2 \simeq \frac{1}{2}\left|\frac{f''(x^*)}{f'(x^*)}\right| \varepsilon_n^2$$

This is OK as long as $f'(x^*) \neq 0$

Quadratic convergence

Fixed point method

Goal: Find x such that $x = g(x)$

If you need to find $f(x) = 0$ Set $g(x) = f(x) + x$

and find x such that $x = g(x)$

This is an iterative method

Start with initial guess x_0

$$x_{n+1} = g(x_n)$$

If $x_n \to x^*$ then $x_{n+1} \to x^*$ and $g(x_n) \to g(x^*)$

then $x^* = g(x^*)$

x_n does not always converge.

$\varepsilon_n = x^* - x_n$

$\varepsilon_{n+1} = x^* - x_{n+1} = g(x^*) - g(x_n) = g'(\xi)(x^* - x_n) = g'(\xi)\,\varepsilon_n$

ξ between x_n and x^*

$|\varepsilon_{n+1}| \simeq |g'(x^*)||\varepsilon_n|$

We will have «local» convergence if $|g'(x^*)| < 1$

local means x_0 is close enough to x^*

the convergence is linear (Obs: if $|g'(\xi)| < 1$ for all

$\}$ then we have global convergence)

Algorithm (Newton)

Input: $f(x)$ and $x = x_0$ and $f'(x)$, ε_x and ε_y errors

Output: root of $f(x)$ in x

$f_0 = f(x)$

$f_1 = f'(x)$

$y = x - \frac{f_0}{f_1}$

$\varepsilon_1 = y - x$

$f_0 = f(y)$

while $(|\varepsilon_1| > \varepsilon_x$ or $|f_0| > \varepsilon_y)$

$x = y$

$f_1 = f'(x)$

$y = x - \frac{f_0}{f_1}$

$\varepsilon_1 = y - x$

$f_0 = f(y)$

end

Algorithm (fixed point)

Input: x and $g(x)$, ε

output: x = root of $x = g(x)$

$y = g(x)$

while $(|y-x| > \varepsilon)$

$\quad x = y$

$\quad y = g(x)$

end