Note Title 3/30/2015

<u>Odes</u> $u' = f(u, t)$

$u(0) = u_0$

$u_n \approx u(nh)$ $h =$ timestep

u_n approximation obtained from a numerical method like Euler, Runge Kutta, etc.

<u>Def</u>: A method is convergent if

$$\lim_{\substack{h \to 0 \\ nh = t}} u_n = u(t),$$

where $u(t)$ is the exact solution to the initial value problem

<u>Def</u>: A method is stable if errors on the initial condition do not get amplified.

<u>Example</u>: Euler's method

$$e_n = u(t_n) - u_n \qquad t_n = nh$$

$$|e_n| \le e^{L(t_n - t_0)} |e_0| + \frac{T}{L}\left(e^{L(t_n-t_0)} - 1\right)$$

$$L = \max \left|\frac{\partial f}{\partial u}\right| \qquad T = \text{bound on } \frac{h}{2}|u''(t)|$$

t fixed $\quad nh = t \quad t_n = t$

$\longrightarrow 0$ as $h \to 0$

$$|u(t) - u_n| \le e^{L(t-t_0)} |u(0) - u_0| + \underbrace{\frac{\tau}{L}}_{O(h)} \left(e^{L(t-t_0)} - 1 \right)$$

If $u(0) - u_0 = 0 \implies |u(t) - u_n| \xrightarrow[n\to\infty]{} 0$ Converges

Errors due to initial conditions remain bounded

$$\underbrace{e^{L(t-t_0)}}_{\text{bounded}} |u_0 - u(0)|$$

bounded because t is fixed. This means the method is stable.

Partial differential equations (PDEs)

These are equations where the unknown ϕ is a function of two or more variables $\phi = \phi(x,y)$, and partial derivatives of ϕ appear in the equation

Ex: $u = u(x,t)$ $\qquad \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

Def: the order of a PDE is the highest order of the derivative involved

Ex: $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ is of second order

Def: A linear 1^{st} order PDE is an equation of the form

$$a(x,y)\frac{\partial\phi}{\partial x}+b(x,y)\frac{\partial\phi}{\partial y}+c(x,y)\,\phi+d(x,y)=0$$

a, b, c and d are known functions of x and y. ϕ is the unknown.

Def: A linear 2^{nd} order PDE is an equation of the form

$$a\frac{\partial^2\phi}{\partial x^2}+b\frac{\partial^2\phi}{\partial x\partial y}+c\frac{\partial^2\phi}{\partial y^2}+d\frac{\partial\phi}{\partial x}+e\frac{\partial\phi}{\partial y}+f\phi+g=0$$

where a, b, c, d, e, f and g are known functions of (x,y)

Ex: $\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 = 1$ is first order non-linear

Ex: $x\phi \frac{\partial^2 \phi}{\partial x^2} + \left(\frac{\partial \phi}{\partial x}\right)^2 \frac{\partial^2 \phi}{\partial y^2} + \phi^2 = 0$ is quasi linear

Def: $\boxed{a \frac{\partial^2 \phi}{\partial x^2} + b \frac{\partial^2 \phi}{\partial x \partial y} + c \frac{\partial^2 \phi}{\partial y^2} + d = 0}$ $(*)$

is a 2nd order quasilinear if a, b, c and d depend only on $x, y, \phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}$

this equation is:

hyperbolic, if $b^2 - 4ac > 0$

parabolic, if $b^2-4ac=0$

elliptic, if $b^2-4ac<0$

Note: An equation may be of one type on some region of the (x,y) plane and of an other type on other regions.

Ex: a) $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$ (Laplace)

$a=c=1 \quad b=0 \Rightarrow b^2-4ac<0 \Rightarrow$ elliptic

b) $\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}=0$ (wave equation)

$a=1 \quad c=-1 \implies b^2-4ac>0 \implies$ hyperbolic

3) $\dfrac{\partial \phi}{\partial t} = \dfrac{\partial^2 \phi}{\partial x^2}$ heat or diffusion equation

$a=0 \quad b=0 \quad c=-1 \implies b^2-4ac=0 \implies$ parabolic

Black-Scholes

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf = 0$$

$a=0 \quad b=0 \quad c=\frac{1}{2}\sigma^2 S^2 \implies b^2-4ac=0 \implies$ parabolic

<u>Heat equation</u> $\dfrac{\partial \phi}{\partial t} = \dfrac{\partial^2 \phi}{\partial x^2}$

need initial and boundary conditions

We want to solve for $0 \le x \le 1$ and $t \ge 0$

(IC) Initial conditions $\phi(x,0) = \phi_0(x)$ $0 \le x \le 1$

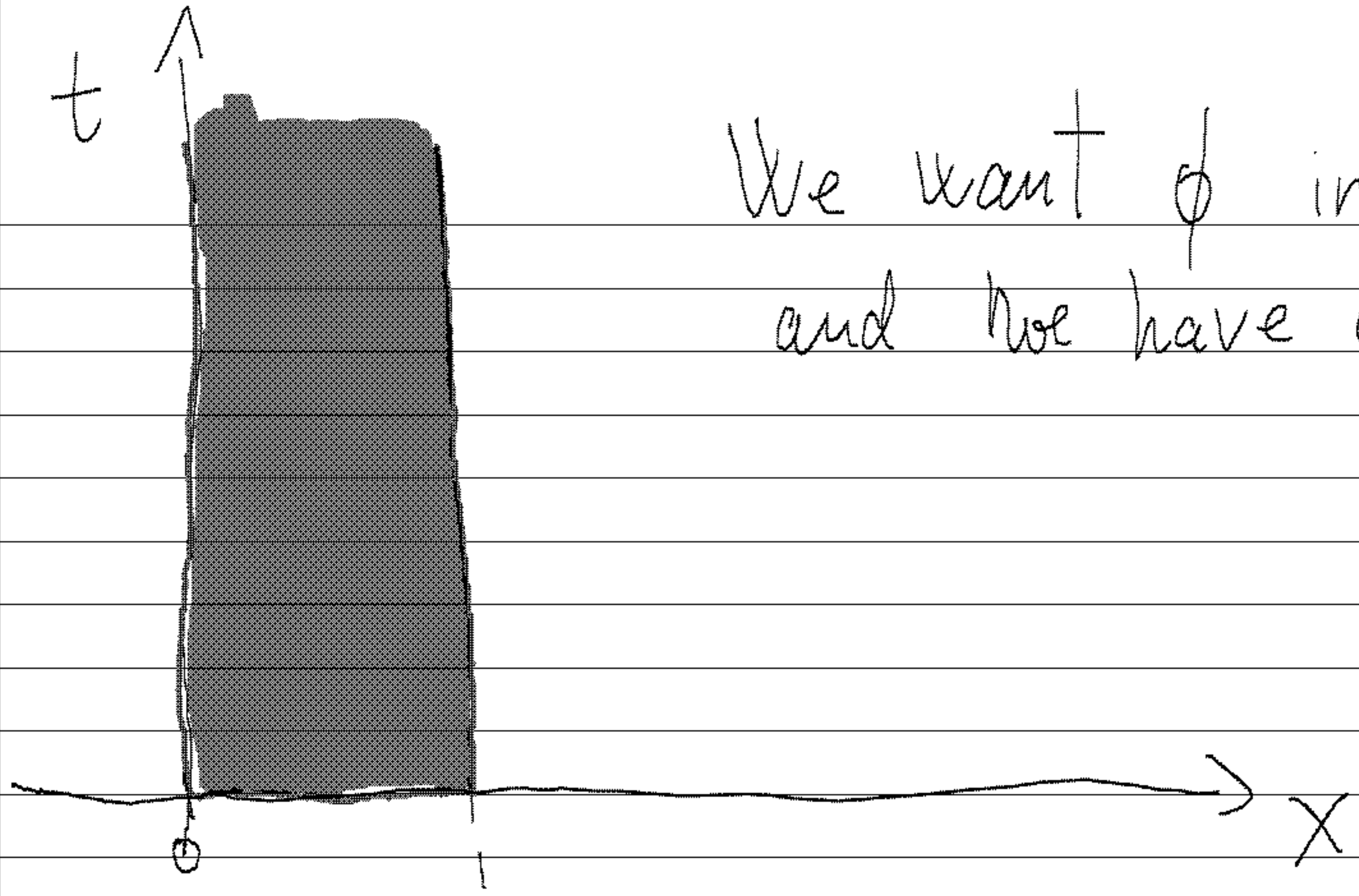
given

(BC) Boundary conditions $\phi(0,t) = \phi_\ell(t)$ $t \ge 0$

$\phi(1,t) = \phi_r(t)$ $t \ge 0$

given

Other boundary conditions are possible



We want ϕ in the shaded region and we have ϕ in the solid border

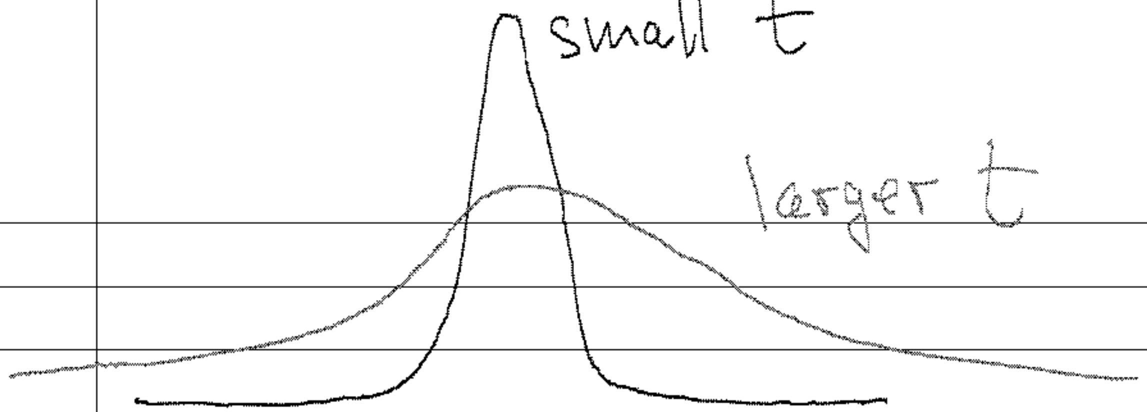
Ex.: $$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

$$\phi(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$$ is a solution for $t>0$ and $-\infty < x < \infty$

$$\frac{\partial \phi}{\partial t} = -\frac{1}{4\sqrt{\pi}\, t^{3/2}} e^{-\frac{x^2}{4t}} + \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \left(\frac{x^2}{4t^2}\right)$$

$$\frac{\partial \phi}{\partial x} = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \left(-\frac{2x}{4t}\right)$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \left(-\frac{2x}{4t}\right)^2 + \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \left(-\frac{2}{4t}\right)$$



Obs: No matter the initial and boundary condition, the solution of the heat equation has ∞ derivatives in t and x in the domain of definition ($t>0$ and $0<x<1$)

Finite difference approximations of first derivatives

Forward: $$\boxed{f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)}$$

Check: $f(x+h) = f(x) + f'(x)h + O(h^2)$

then $\dfrac{f(x+h) - f(x)}{h} = f'(x) + O(h)$ ✓

Backward: $\boxed{f'(x) = \dfrac{f(x) - f(x-h)}{h} + O(h)}$

Check: $f(x-h) = f(x) + f'(x)(-h) + O(h^2)$

$\dfrac{f(x-h) - f(x)}{-h} = f'(x) + O(h)$ ✓