Note Title 3/25/2015

Reminder

Richardson extrapolation

$$T(h) = a_0 + a_1 h^{p_1} + a_2 h^{p_2} + \cdots$$

$$0 < p_1 < p_2 < \cdots$$

$$A_{00} = T(h)$$

$$A_{10} = T(h/2) \quad A_{01} = A_{10} + \frac{A_{10} - A_{00}}{2^{p_1} - 1}$$

$$A_{20} = T(h/4) \quad A_{11} = A_{20} + (A_{20} - A_{10})/(2^{p_1} - 1) \quad A_{02} = A_{11} + (A_{11} - A_{01})/(2^{p_2} - 1)$$

$$A_{30} = T(h/8) \quad A_{21} = A_{30} + (A_{30} - A_{20})/(2^{p_1} - 1) \quad A_{12} = A_{21} + (A_{21} - A_{11})/(2^{p_2} - 1)$$

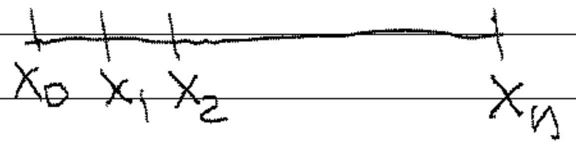
<u>Fact</u>: Let $y(x,h)$ be the solution of $y' = f(x,y)$ and $y(x_0) = y_0$ with Euler's method with step h. Let $y(x)$ be the exact solution then

$$y(x,h) = y(x) + c_1(x)\,h + c_2(x)\,h^2 + \cdots\cdots$$

$$p_1 = 1, \quad p_2 = 2, \quad p_3 = 3, \ldots\ldots$$

We can use Richardson extrapolation

<u>Passive extrapolation</u>



Get $y(x_n, h)$ and then extrapolate

$y(x_n, h/2)$

$\vdots$

$y(x_n, h/2^k)$

<u>Active extrapolation</u>

Get $y(x_1, h)$ extrapolate to get a better approximation

$y(x_1, h/2)$ of $y(x_1)$. Call this approximation y_1

$\vdots$ Star again with initial condition $y(x_1) = y_1$

$y(x_1, h/2^k)$ We do n extrapolations instead of 1.

Note: Varying steps.

First order vector valued odes

$$y = y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix} \qquad F = F(x, y) = \begin{bmatrix} f_1(x, y_1, \dots, y_n) \\ f_2(x, y_1, \dots, y_n) \\ \vdots \\ f_n(x, y_1, \dots, y_n) \end{bmatrix}$$

$y' = F(x,y)$ or $y_1' = f_1(x, y_1, \dots, y_n)$ I.c. $y(x_0) = y_0$

$y_2' = f_2(x, y_1, \dots, y_n)$

vectors

$\vdots$

$$\vdots$$

$$y_n' = f_n(x, y_1, \ldots, y_n)$$

<u>Example</u> $y_1' = 3y_1 + y_2$ $\qquad \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 3y_1 + y_2 \\ y_1 - y_2 \end{bmatrix}$

$y_2' = y_1 - y_2$

<u>Obs</u>: Euler, Improved Euler and Runge-Kutta can all be used to solve vector valued odes.

<u>Second order odes</u>

$(*)$ $y'' = f(x, y, y')$

Set $z_1 = y$ and $z_2 = y'$ then (*) is equivalent to

$$z_1' = z_2$$
$$z_2' = f(x, z_1, z_2)$$

We transformed the second order equation to a system of two first order equations

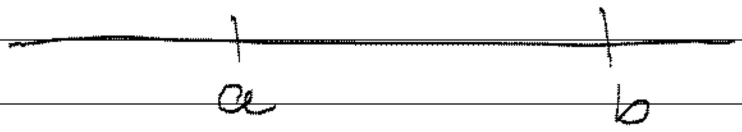
$$z' = F(x, z) \qquad F = \begin{bmatrix} z_2 \\ f(x, z_1, z_2) \end{bmatrix}$$

If the initial condition was $y(x_0) = y_0$ and $y'(x_0) = y_1$, now it is $z(x_0) = \begin{bmatrix} z_1(x_0) \\ z_2(x_0) \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$

Boundary value problems

$y'' = f(x, y, y')$ $\quad y(a) = A$ and $y(b) = B$

Find $y(x)$ for $a \le x \le b$



Auxiliary problem

$y(x,s)$ $\quad y' = \frac{\partial y}{\partial x}$

$y'' = f(x, y, y')$

$y(a) = A$

$y'(a) = s$

Let $\phi(s) = y(b, s) - B$

Find s^* zero of ϕ

Then $y(x, s^*)$ is the solution to the boundary value problem

Two options to find the zero of $\phi(s)$

1) Bisection. Need to be able to compute $\phi(s)$.

2) Newton-Rapson. $s_{n+1} = s_n - \dfrac{\phi(s_n)}{\phi'(s_n)}$

Need to be able to evaluate $\phi'(s)$

$y'' = f(x, y, y')$ $\quad y(a) = A$ and $y'(a) = s$

Change to system

$$(*) \quad \begin{cases} z_1' = z_2 & z_1(a) = A \\ z_2' = f(x, z_1, z_2) & z_2(a) = s \end{cases}$$

Take derivatives of (*) with respect to s

(**) $$\begin{cases} \left(\dfrac{\partial z_1}{\partial s}\right)' = \dfrac{\partial z_2}{\partial s} & \dfrac{\partial z_1}{\partial s}(a) = 0 \\ \left(\dfrac{\partial z_2}{\partial s}\right)' = \dfrac{\partial f}{\partial z_1}(x, z_1, z_2)\dfrac{\partial z_1}{\partial s} + \dfrac{\partial f}{\partial z_2}(x, z_1, z_2)\dfrac{\partial z_2}{\partial s} & \dfrac{\partial z_2}{\partial s}(a) = 1 \end{cases}$$

Let $z_3 = \dfrac{\partial z_1}{\partial s}$ and $z_4 = \dfrac{\partial z_2}{\partial s}$

Eqs $$\begin{cases} z_1' = z_2 \\ z_2' = f(x, z_1, z_2) \end{cases}$$

$$\begin{cases} z_3' = z_4 \\ z_4' = \dfrac{\partial f}{\partial z_1}(x, z_1, z_2)\, z_3 + \dfrac{\partial f}{\partial z_2}(x, z_1, z_2)\, z_4 \end{cases}$$

$$\text{IC} \begin{cases} z_1(a) = A \\ z_2(a) = s \\ z_3(a) = 0 \\ z_4(a) = 1 \end{cases}$$

We solve Eqs + IC all 4 simultaneously

$\phi(s) = z_1(b) - B$ $\qquad$ $\phi'(s) = \dfrac{\partial y}{\partial s}(b, s) = \dfrac{\partial z_1}{\partial s}(b) = z_3(b)$

$\phi'(s) = z_3(b)$

$$s_{n+1} = s_n - \frac{(z_1(b) - B)}{z_3(b)}$$