Note Title 3/23/2015

Ordinary differential equations

First order $\boxed{y' = f(x,y)}$ (1)

f is known. y is the unknown. $y = y(x)$. $y' = \frac{dy}{dx}$

Initial conditions $\boxed{y(x_0) = y_0}$ (2) x_0 and y_0 known

(1) and (2) together are called initial value problem.

Linear equations

$$y' + a(x)\,y = b(x)$$

$$I(x) = e^{\int a(x)dx}$$

Multiply the equation by $I(x)$ to get

$$y' e^{\int a(x)dx} + e^{\int a(x)dx} a(x) y = b(x) e^{\int a(x)dx}$$

$$\frac{d}{dx}\left[y(x)\, e^{\int a(x)dx}\right] = b(x)\, e^{\int a(x)dx}$$ Integrate both sides

$$y(x)\, e^{\int a(x)dx} = \int b(x)\, e^{\int a(x)dx}\, dx + C$$

$$y(x) = e^{-\int a(x)dx} \left\{ \int b(x)\, e^{\int a(x)dx}\, dx + C \right\}$$

<u>Example</u>

$$y' + \frac{1}{x} y = 1$$
$$y(1) = 1$$

$$I = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$xy' + y = x$$

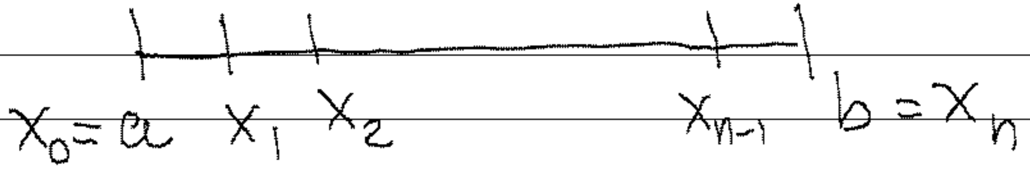
$$\frac{d}{dx}(xy) = x \qquad xy = \frac{x^2}{2} + C \qquad y = \frac{x}{2} + \frac{C}{x}$$

$$y(1) = 1 \implies 1 = \frac{1}{2} + C \implies C = \frac{1}{2} \qquad \boxed{y = \frac{x}{2} + \frac{1}{2x}}$$

One-step methods

We seek $y(x)$, the solution of

$$y' = f(x,y)$$
$$y(x_0) = y_0 \qquad \text{for } a = x_0 \leq x \leq b$$



$$h = \frac{b-a}{n} \qquad x_k = a + kh$$

$$0 \le k \le n$$

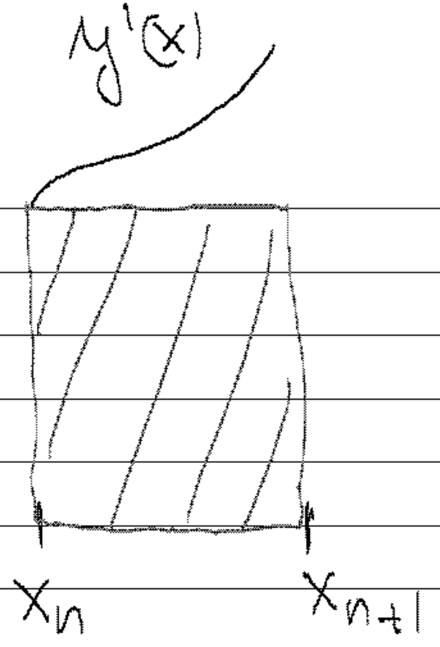
$$y_n \approx y(x_n)$$

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} y'(x)\,dx$$

$$I_n \approx \int_{x_n}^{x_{n+1}} y'(x)\,dx$$

$$y_{n+1} = y_n + I_n$$

<u>Euler's method</u>



Red area $= h\, y'(x_n) \cong h\, f(x_n, y_n)$

$I_n = h f(x_n, y_n)$

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

$$\boxed{y_{n+1} = y_n + (x_{n+1} - x_n)\, f(x_n, y_n)}$$

Error analysis

$e_n = y(x_n) - y_n$

$y(x_{n+1}) = y(x_n) + y'(x_n)\, h + \frac{1}{2}\, y''(\xi_n)\, h^2$

where $\xi_n \in (x_n, x_{n+1})$

$$e_{n+1} = y(x_{n+1}) - y_{n+1} = y(x_n) + y'(x_n)h + \frac{1}{2} y''(\xi_n) h^2 - (y_n + hf(x_n, y_n))$$

$$= e_n + h\left[f(x_n, y(x_n)) - f(x_n, y_n)\right] + \frac{1}{2} h^2 y''(\xi_n)$$

$$= e_n + h \frac{\partial f}{\partial y}(x_n, \beta_n)\, e_n + \frac{1}{2} h^2 y''(\xi_n)$$

for some β_n between $y(x_n)$ and y_n

$$e_{n+1} = \left[1 + h \frac{\partial f}{\partial y}(x_n, \beta_n)\right] e_n + \frac{1}{2} h^2 y''(\xi_n)$$

Thus, $|e_{n+1}| \le (1+hL)\,|e_n| + hT$

where L is a bound of $\left|\frac{\partial f}{\partial y}\right|$ and T is a bound of

$\frac{1}{2} h |y''(\xi)|$

Difference equations $\quad a_{n+1} - \alpha a_n = \gamma \quad$ α & γ are given

goal: Find a_n for all n.

$$a_1 = \alpha a_0 + \gamma$$

$$a_2 = \alpha a_1 + \gamma = \alpha^2 a_0 + \alpha\gamma + \gamma$$

$$a_3 = \alpha^3 a_0 + \alpha^2 \gamma + \alpha\gamma + \gamma$$

$$a_n = \alpha^n a_0 + \gamma \left(\alpha^{n-1} + \alpha^{n-2} + \dots + \alpha + 1\right)$$

$$a_n = \alpha^n a_0 + \gamma \frac{\alpha^n - 1}{\alpha - 1}$$

Error $|e_{n+1}| \le (1+hL)|e_n| + hT$

$$|e_n| \le (1+hL)^n |e_0| + T \frac{(1+hL)^n - 1}{L}$$

$$1+hL \le e^{hL}$$

$$|e_n| \le e^{nhL} |e_0| + \frac{T}{L}\left(e^{nhL} - 1\right)$$

$$nh = x_n - x_0$$

$$|e_n| \le e^{L(x_n - x_0)} |e_0| + \left(\frac{T}{L}\right)\left(e^{L(x_n - x_0)} - 1\right)$$

↑ error at x_n, $e_n = y(x_n) - y_n$

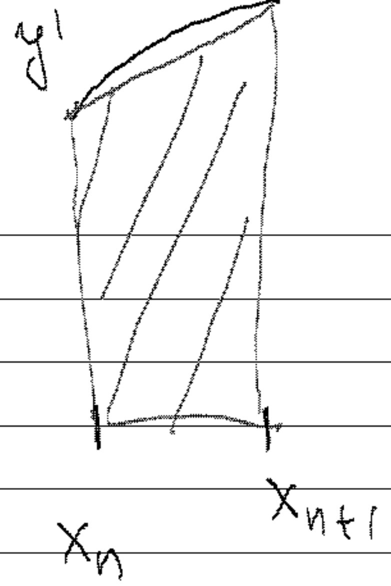
where L = bound on $\left|\frac{\partial f}{\partial y}\right|$

T = bound on $\frac{1}{2} h \left|y''(x)\right|$

Note: $T = O(h)$

Roughly error $= O(|e_0| + h)$

<u>Trapezoidal rule</u>



$$I_n = \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

$$\boxed{y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]} \quad (*)$$

This is an implicit method because y_{n+1} appears in the right hand side. To get y_{n+1} we need to find the root of a non-linear equation, i.e. use bisection or Newton-Rapson.

Advantage: It is more accurate than Euler's

Disadvantage: Need to find the zero of a non-linear function at each step.

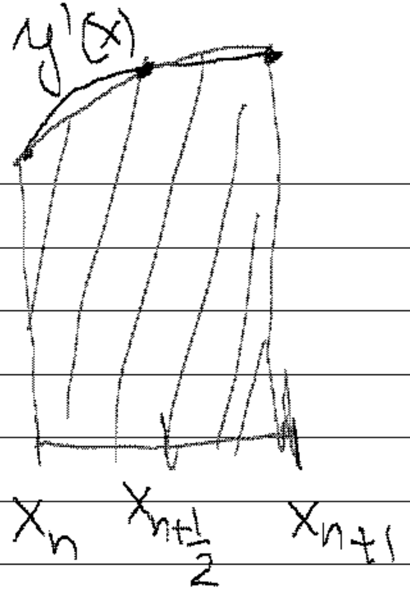
Improved Euler's method

$$\boxed{y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + h f(x_n, y_n))\right]}$$

we used $y_n + h f(x_n, y_n)$ as an approximation to y_{n+1} in the right hand side of (*)

Runge-Kutta

$$I_n = \frac{h}{6}\left[y'(x_n) + 4\, y'(x_{n+\frac{1}{2}}) + y'(x_{n+1})\right]$$



$$y'(x_n) \approx f(x_n, y_n)$$

$$x_{n+1/2} = \frac{x_n + x_{n+1}}{2}$$

$$y'(x_{n+1/2}) \approx f(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) \qquad y'(x_{n+1}) \approx f(x_{n+1}, y_{n+1})$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_1\right) \approx f(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}})$$

$$k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_2\right) \approx f(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}})$$

$$k_4 = f(x_{n+1}, y_n + h k_3) \approx f(x_{n+1}, y_{n+1})$$

$$y_{n+1} = y_n + \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$\text{Error} = O(h^2)$$