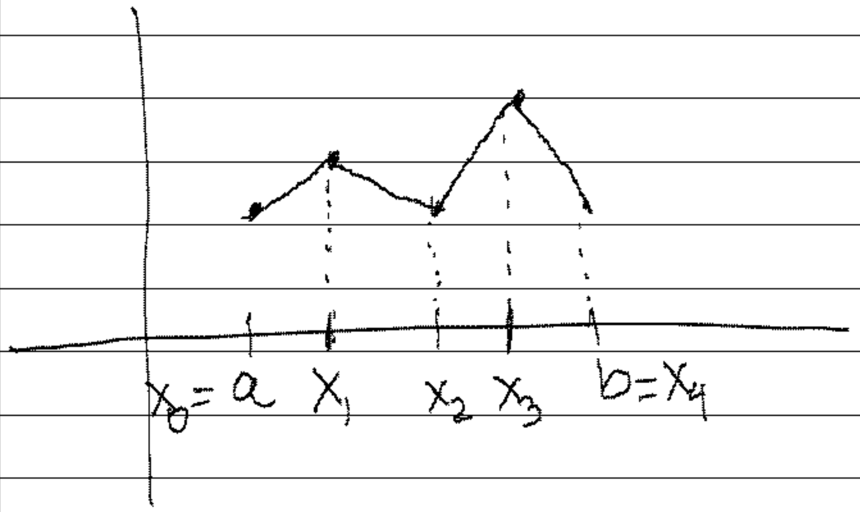
Splines d = degree

If $d=1$ This gives a continuous piecewise linear function



(x_i, y_i) $0 \le i \le n$ given

$x_0 = a$ $x_n = b$

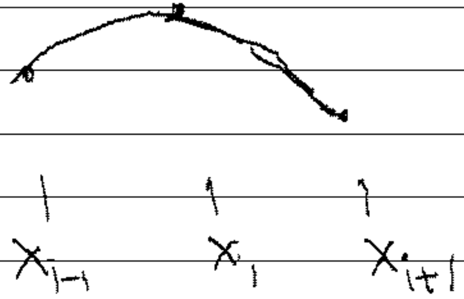
Interpolate with $s(x)$.

$s(x)$ is a polynomial of degree d within $[x_{i-1}, x_i]$ for all i.

also $s(x), s'(x), \ldots, s^{(d-1)}(x)$ is continuous

$\underline{d=1}$: $$s(x) = y_i \frac{(x - x_{i+1})}{(x_i - x_{i+1})} + y_{i+1} \frac{(x - x_i)}{x_{i+1} - x_i} \qquad x \in [x_i, x_{i+1}]$$

$\underline{d=3}$: Cubic splines $$s(x) = P_i(x) \qquad x \in [x_{i-1}, x_i]$$



<u>Equations to be satisfied</u>

$$\left.\begin{array}{ll} 1)\ P_1(x_0) = y_0 & \\ \quad P_i(x_i) = P_{i+1}(x_i) = y_i & 1 \le i \le n-1 \end{array}\right\} \quad s(x_i) = y_i$$

$P_n(x_n) = y_n$

2) $P_i'(x_i) = P_{i+1}'(x_i)$ $\quad 1 \le i \le n-1$ $\quad \}$ $S'(x)$ is continuous

3) $P_i''(x_i) = P_{i+1}''(x_i)$ $\quad 1 \le i \le n-1$ $\quad \}$ $S''(x)$ is continuous

$S''(x_i) = P_i''(x_i) = P_{i+1}''(x_i) = M_i$

$$P_i''(x) = M_i \left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right) + M_{i-1}\left(\frac{x - x_i}{x_{i-1} - x_i}\right) \qquad x \in [x_{i-1}, x_i]$$

Integrate Twice

$$P_i(x) = \frac{M_i}{6}\frac{(x - x_{i-1})^3}{(x_i - x_{i-1})} + \frac{M_{i-1}}{6}\frac{(x - x_i)^3}{(x_{i-1} - x_i)} + c_i x + d_i$$

$P_i(x_i) = y_i$ and $P_i(x_{i-1}) = y_{i-1}$ imply

$$\frac{M_i}{6}(x_i - x_{i-1})^2 + c_i x_i + d_i = y_i \qquad \text{Eq (1)}$$

$$\frac{M_{i-1}}{6}(x_{i-1} - x_i)^2 + c_i x_{i-1} + d_i = y_{i-1} \qquad \text{Eq (2)}$$

Substract and devide by $x_i - x_{i-1}$ to get

$$c_i = \frac{y_i - y_{i-1}}{x_i - x_{i-1}} - \frac{(x_i - x_{i-1})}{6}(M_i - M_{i-1})$$

x_{i-1} times Eq(1) $-$ x_i times Eq(2)

$$-(x_i - x_{i-1})\, d_i + \frac{(x_i - x_{i-1})^2}{6}(x_{i-1} M_i - x_i M_{i-1}) = x_{i-1} y_i - x_i y_{i-1}$$

$$d_i = \frac{(x_i - x_{i-1})}{6}(M_i x_{i-1} - M_{i-1} x_i) - \frac{(x_{i-1} y_i - x_i y_{i-1})}{(x_i - x_{i-1})}$$

$$c_i x + d_i = \frac{(x - x_{i-1})}{(x_i - x_{i-1})} y_i + \frac{(x_i - x)}{(x_i - x_{i-1})} y_{i-1} - \frac{(x_i - x_{i-1})}{6} \left\{ M_i (x - x_{i-1}) + M_{i-1}(x_i - x) \right\}$$

$$\boxed{\begin{aligned} P_i(x) = {} & \frac{M_i}{6} \frac{(x - x_{i-1})}{(x_i - x_{i-1})} \left\{ (x - x_{i-1})^2 - (x_i - x_{i-1})^2 \right\} + \frac{M_{i-1}}{6} \frac{(x_i - x)}{(x_i - x_{i-1})} \left\{ (x_i - x)^2 - (x_i - x_{i-1})^2 \right\} \\ & + \frac{(x - x_{i-1})}{(x_i - x_{i-1})} y_i + \frac{(x_i - x)}{(x_i - x_{i-1})} y_{i-1} \qquad \text{for} \quad x \in [x_{i-1}, x_i] \end{aligned}}$$

$$P'_i(x_i) = P'_{i+1}(x_i) \qquad \text{for} \quad 1 \le i \le n-1$$

$$\frac{M_i}{6}\,\frac{\left[3(x_i-x_{i-1})^2-(x_i-x_{i-1})^2\right]}{(x_i-x_{i-1})}+\frac{M_{i-1}}{6}\,\frac{(x_i-x_{i-1})^2}{(x_i-x_{i-1})}+\frac{y_i-y_{i-1}}{(x_i-x_{i-1})}=$$

$$\frac{M_{i+1}}{6}\left[-\frac{(x_{i+1}-x_i)^2}{(x_{i+1}-x_i)}\right]+\frac{M_i}{6}\,\frac{\left[-3(x_{i+1}-x_i)^2+(x_{i+1}-x_i)^2\right]}{(x_{i+1}-x_i)}+\frac{y_{i+1}-y_i}{x_{i+1}-x_i}$$

$$\boxed{\frac{(x_i-x_{i-1})}{6}M_{i-1}+\left[\frac{(x_i-x_{i-1})+(x_{i+1}-x_i)}{3}\right]M_i+\frac{(x_{i+1}-x_i)}{6}M_{i+1}=\frac{y_{i+1}-y_i}{x_{i+1}-x_i}-\frac{y_i-y_{i-1}}{x_i-x_{i-1}}}$$

$\boxed{1\le i\le n-1}$

<u>Obs</u>: We have $n-1$ equations for $n+1$ unknowns $(M_0, M_1, \dots, M_n)$

Need 2 more equations

<u>Options</u>: 1) $\boxed{M_0 = M_n = 0}$

2) If $f'(a)$ and $f'(b)$ are known, $P_1'(x_0) = f'(a)$ and $P_n'(x_n) = f'(b)$

$$P_1'(x_0) = -\frac{M_0}{2}(x_1 - x_0) + \frac{y_1 - y_0}{x_1 - x_0} - \frac{(x_1 - x_0)}{6}(M_1 - M_0)$$

$$P_n'(x_n) = \frac{M_n}{2}(x_n - x_{n-1}) + \frac{y_n - y_{n-1}}{x_n - x_{n-1}} - \frac{(x_n - x_{n-1})}{6}(M_n - M_{n-1})$$

$$\boxed{\begin{array}{l} \frac{(x_1 - x_0)}{3} M_0 + \frac{(x_1 - x_0)}{6} M_1 = -f'(a) + \frac{y_1 - y_0}{x_1 - x_0} \\ \hline \frac{(x_n - x_{n-1})}{6} M_{n-1} + \frac{(x_n - x_{n-1})}{3} M_n = f'(b) - \frac{y_n - y_{n-1}}{x_n - x_{n-1}} \end{array}}$$

Matrix if $x_0, x_1, \ldots, x_n$ are equidistant and $\Delta = x_{i+1} - x_i$

$$\Delta \begin{bmatrix} 2/3 & 1/6 & 0 & \cdots & 0 \\ 1/6 & 2/3 & 1/6 & & \vdots \\ 0 & 1/6 & 2/3 & 1/6 & \\ \vdots & \ddots & \ddots & \ddots & 0 \\ & & 1/6 & 2/3 & 1/6 \\ 0 & \cdots & 0 & 1/6 & 2/3 \end{bmatrix}$$

$x^T A x \geq 0$

Use Cholesky

<u>Nonlinear equations</u>

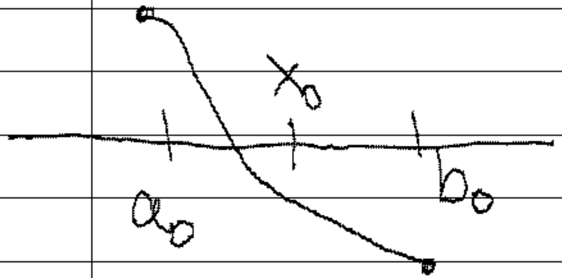
$f: \mathbb{R} \to \mathbb{R}$ continuous

Goal: Find x such that $f(x) = 0$

Bisection

Start with $a_0 < b_0$ such that $f(a_0) f(b_0) < 0$

We know there is x^* such that $a_0 < x^* < b_0$ and $f(x^*) = 0$



Set $x_0 = \dfrac{a_0 + b_0}{2}$.

If $f(a_0) f(x_0) < 0$, f has a zero in (a_0, x_0), so set $a_1 = a_0$ and $b_1 = x_0$

Otherwise, set $a_1 = x_0$ and $b_1 = b_0$.

We know f has a zero in (a_1, b_1)

Given a_i and b_i such that $f(a_i)f(b_i) < 0$, set $x_i = \frac{a_i + b_i}{2}$.

If $f(a_i)f(x_i) < 0$ then $a_{i+1} = a_i$ and $b_{i+1} = x_i$

Otherwise $a_{i+1} = x_i$ and $b_{i+1} = b_i$

Algorithm (bisection)

Input: f, a and b such that $f(a)f(b) < 0$, ε

while $(b - a > 2\varepsilon)$

$x = \frac{a+b}{2}$

If $f(x)f(a) < 0$

```
        b = x
    else
        a = x
    end
end
x = (a+b)/2
```

this method always converges. It is iterative. If x^* is the real solution, and x_k the k^{th} estimate the method gives

$$|x_{k+1} - x^*| \approx \frac{1}{2} |x_k - x^*|$$ Linear convergence.