Note Title 3/11/2015

Example: Analytical solutions of Black-Scholes

For European calls

C = value of the call at time $t=0$

S = value of the asset at time $t=0$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\log(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$\boxed{C = S\,N(d_1) - K e^{-rT} N(d_2)}$$

For European put

P = value of the put at $t=0$

$$P = K e^{-rT} N(-d_2) - S\,N(-d_1)$$

Assume we have C but not σ, and we want to find σ ("implied volatility")

$d_1 = d_1(\sigma)$ and $d_2 = d_2(\sigma)$

$$f(\sigma) = C - S\,N(d_1(\sigma)) + K e^{-rT} N(d_2(\sigma))$$

Our goal is to find σ such that $f(\sigma) = 0$

We can use bisection or NR.

$$N(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-y^2/2}\, dy = \frac{1}{2}$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-y^2/2}\, dy + \frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{-y^2/2}\, dy$$

$$= \frac{1}{2} + \begin{cases} \frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{-y^2/2}\, dy & \text{if } x > 0 \\ \frac{-1}{\sqrt{2\pi}} \int_{x}^{0} e^{-y^2/2}\, dy & \text{if } x < 0 \end{cases}$$

We can use Trapezoidal, with or without Richardson extrapolation, we could use Simpson to compute $N(x)$

Gaussian quadrature

I is an interval, $I = (-\infty, a]$ or $[a,b]$ or $[b, \infty)$

$w : I \to \mathbb{R}$ $\quad w(x) \geq 0$ for all $x \in I$

Goal: Compute $\int_I w(x) f(x)\,dx$,

We want a formula of the form

$$\int_I w(x) f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

Questions: 1) How to pick x_i s?

2) How to pick the w_i s?

Answer to 1) Do not pick the x_is equidistant.

Follow this theory

<u>Gaussian</u> <u>quadrature</u>

Construct a family of polynomials P_k $0 \leq k \leq n$ that satisfy the following properties: a) degree of $P_k = k$ and

b) $$\int_I w(x) P_k(x) P_j(x)\, dx = \begin{cases} 0 & \text{if } k \neq j \\ 1 & \text{if } k = j \end{cases}$$

$\boxed{\text{Fact 1)}}$ these polynomials can be constructed using the Gram-Schmidt algorithm

$\boxed{\text{Fact 2)}}$ $P_n(x)$ has n zeros in I

Select $x_1, x_2, \ldots, x_n$ to be the zeros of $P_n(x)$

<u>Gram - Schmidt</u>

Define $\langle f, g \rangle = \int_I w(x) f(x) g(x)\, dx$

$$P_0 = \frac{1}{\sqrt{\langle 1,1 \rangle}} \qquad \langle 1,1 \rangle = \int_I w(x)\, dx$$

Once we have $P_0, P_1, \ldots, P_{k-1}$ we P_k as follows

$$Q_k(x) = x^k - \sum_{j=0}^{k-1} \langle x^k, P_j \rangle P_j$$

$$P_k(x) = \frac{Q_k(x)}{}$$

$\sqrt{\langle Q_k, Q_k \rangle}$

Why this works? $\langle \, , \rangle$ is an inner product, i.e.

1) $\langle f, g \rangle = \langle g, f \rangle$

2) $\langle af+bg, h \rangle = a\langle f, h \rangle + b\langle g, h \rangle$

3) $\langle f, f \rangle \geq 0$ & $\langle f, f \rangle = 0 \iff f = 0$

Obs: $\left\langle \frac{f}{\sqrt{\langle f,f \rangle}}, \frac{f}{\sqrt{\langle f,f \rangle}} \right\rangle = \frac{\langle f,f \rangle}{(\sqrt{\langle f,f \rangle})^2} = 1$

$\langle Q_1, P_0 \rangle = \langle X - \langle X, P_0 \rangle P_0, P_0 \rangle = \langle X, P_0 \rangle - \langle \langle X, P_0 \rangle P_0, P_0 \rangle$

$= \langle X, P_0 \rangle - \langle X_0, P_0 \rangle \underbrace{\langle P_0, P_0 \rangle}_{=1} = 0$

If $k < j$

$$\langle Q_k, P_j \rangle = \langle x^k - \sum_{l=0}^{k-1} \langle x^k, P_l \rangle P_l , P_j \rangle =$$

$$= \langle x^k, P_j \rangle - \sum_{l=0}^{k-1} \langle x^k, P_l \rangle \underbrace{\langle P_l, P_j \rangle}_{\substack{=0 \text{ if } l \neq j \\ =1 \text{ if } l = j}} = \langle x^k, P_j \rangle -$$

$$- \langle x^k, P_j \rangle = 0$$

Back to the goal

$$(*) \quad \int_I w(x) f(x) dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

1) Get $x_1, \dots, x_n$ (we did it)

2) Get $w_1, \dots, w_n$

Pick $w_1, \ldots, w_n$ so that $(*)$ is exact for $f(x) =$
$P_0(x), \; P_1(x), \ldots, P_{n-1}(x)$

Note: If $k \geq 1$ $\boxed{\int_I w(x) P_k(x)\, dx = \langle P_k, 1 \rangle = \sqrt{\langle 1,1 \rangle} \langle P_k, P_0 \rangle = 0}$

$$P_0 = \frac{1}{\sqrt{\langle 1,1 \rangle}}$$

In $(*)$ replace f by P_0 to get

(1st) $\boxed{w_1 + w_2 + \cdots + w_n = \int_I w(x)\, dx}$

In $(*)$ replace f by P_k

$1 \le k \le n-1$

$(k+1)^{th})$ $$\boxed{w_1 P_k(x_1) + w_2 P_k(x_2) + \cdots + w_n P_k(x_n) = 0}$$

<u>Example</u> $I = (-\infty, \infty)$

$w(x) = \dfrac{e^{-x^2}}{\sqrt{\pi}}$ $\quad \dfrac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2}\, dx = 1$ $\quad \langle 1, 1 \rangle = 1$

$$\boxed{P_0(x) = 1}$$

$k=1$ $\quad Q_1(x) = x - \langle x, P_0(x) \rangle P_0$

$\langle x, P_0(x) \rangle = \dfrac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2}\, x\, dx = 0$

$$\boxed{P_1(x) = \frac{x}{\sqrt{\langle x, x\rangle}} = \sqrt{2}\,x}$$

$$\langle x, x\rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} x^2\, dx = \frac{1}{2}$$

$k=2$

$$Q_2(x) = x^2 - \langle x^2, P_0\rangle P_0 - \langle x^2, P_1\rangle P_1$$

$$\langle x^2, P_0\rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} x^2\, dx = \frac{1}{2}$$

$$\langle x^2, P_1\rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} x^2 \sqrt{2}\, x\, dx = 0$$

$$Q_2(x) = x^2 - \frac{1}{2}$$

$$P_2(x) = \frac{Q_2(x)}{\sqrt{\langle Q_2, Q_2 \rangle}} = \sqrt{2}\, x^2 - \frac{1}{\sqrt{2}}$$

$$\boxed{x_1 = -\frac{1}{\sqrt{2}} \qquad x_2 = \frac{1}{\sqrt{2}}}$$

Getting w_1 & w_2

$$\boxed{w_1 + w_2 = 1}$$

$$w_1 P_1(x_1) + w_2 P_1(x_2) = 0$$

$$w_1 \sqrt{2}\, x_1 + w_2 \sqrt{2}\, x_2 = 0$$

$\boxed{-w_1 + w_2 = 0}$ then $w_1 = w_2 = \frac{1}{2}$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx \approx \frac{1}{2} f\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{2} f\left(\frac{1}{\sqrt{2}}\right)$$

Obs: The formula $\int_I w(x) f(x)\, dx \approx \sum_{i=1}^{n} w_i f(x_i)$

is exact for all polynomials of degree $\leq 2n-1$