Note Title 2/9/2015

Cholesky factorization: $A = U^T U$, if A is symmetric positive definite. U is upper triangular

Example: $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 5 & -2 \\ 0 & -2 & 2 \end{bmatrix}$

$u_{11} = \sqrt{a_{11}} = 1$

$u_{12} = \dfrac{a_{12}}{u_{11}} = 1$

$u_{22} = \sqrt{a_{22} - u_{12}^2} = 2$

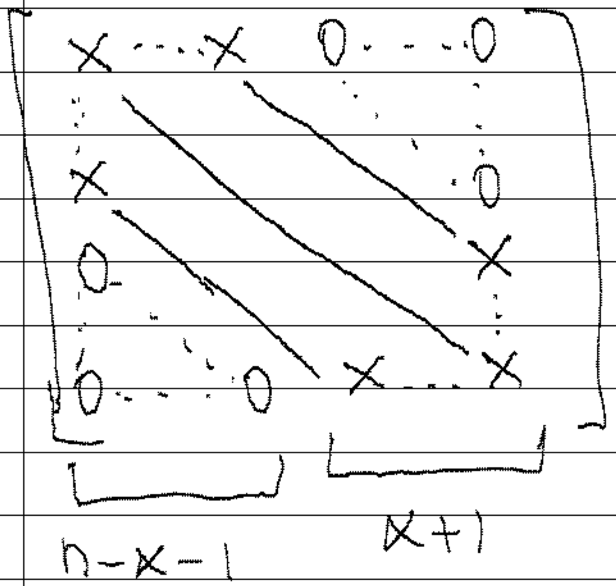
$u_{13} = \dfrac{a_{13}}{u_{11}} = 0$

$u_{23} = \dfrac{a_{23} - u_{12} u_{13}}{u_{22}} = \dfrac{-2 - 1(0)}{2} = -1$

$u_{33} = \sqrt{a_{33} - u_{13}^2 - u_{23}^2} = \sqrt{2 - 0^2 - (-1)^2} = 1$

$$U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \qquad U^T U = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 5 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

<u>Banded</u> <u>matrices</u> α = bandwidth of the matrix



<u>Obs</u>: A is symmetric positive definite of bandwidth α

$A = U^T U$ the Cholesky factorization, then U has bandwidth α.

Algorithm

for $j = 1:n$ $\qquad 1 + 2 + \cdots + j - \beta \cong \frac{(j-\beta)^2}{2} \simeq \frac{\alpha^2}{2}$

$\quad$ $\beta = \max\{1, j - \alpha\}$

$\quad$ for $i = \beta, j-1$

$\qquad$ $u_{ij} = a_{ij}$

$\frac{\alpha^2}{2} + \alpha$ $\qquad$ for $k = \beta : i-1$

$\qquad\quad$ $u_{ij} = u_{ij} - u_{ki} u_{kj}$ $\quad \} 1 \;\} \; i - \beta$

```
        end
        u_ij = u_ij / u_ii                    } 1
    end
    u_jj = a_jj
    for k = β : j-1
        u_jj = u_jj - u_kj^2                  } 1   } j-β = α
    end
    u_jj = √u_jj                              } 1
end
```

<u>Complexity</u>: $\left(\frac{\alpha^2}{2}+\alpha\right)n = O(\alpha^2 n)$

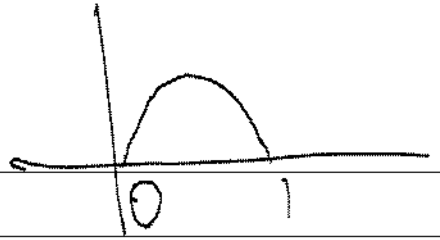
<u>Obs</u>: $A = U^T U$, with U upper triangular, to solve $Ax=b$, first solve $U^T y = b$, then solve $Ux = y$. then

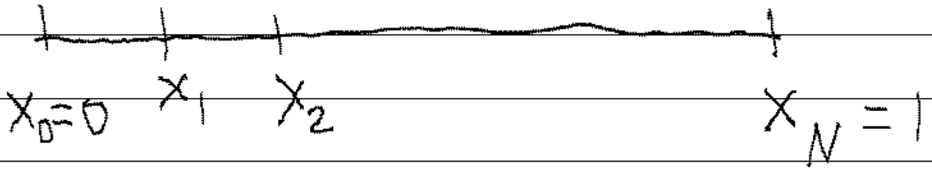
$Ax = U^T U x = U^T y = b$

Solving $U^T y = b$ is easy, with forward substitution. Complexity $= O(n^2)$. If U has bandwidth α, the complexity is $\simeq \alpha n = O(\alpha n)$

<u>Example</u>: $-y'' = 1 \qquad 0 \le x \le 1 \qquad y(0) = y(1) = 0$

$$y'' = \frac{d^2y}{dx^2} \qquad y = -\frac{x^2}{2} + \frac{x}{2}$$





$$x_i = \frac{i}{N} \qquad \Delta x = \frac{1}{N}$$

$$0 \le i \le N$$

$$y''(x_i) \approx \frac{y'\left(x_i + \frac{\Delta x}{2}\right) - y'\left(x_i - \frac{\Delta x}{2}\right)}{\Delta x}$$

$$y'\left(x_i + \frac{\Delta x}{2}\right) \approx \frac{y(x_i + \Delta x) - y(x_i)}{\Delta x} \qquad y_i \approx y(x_i)$$

$$y'\left(x_i - \frac{\Delta x}{2}\right) \approx \frac{y(x_i) - y(x_i - \Delta x)}{\Delta x}$$

$$y''(x_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2}$$

$$-y_{i-1} + 2y_i - y_{i+1} = (\Delta x)^2 \qquad 1 \le i \le N-1 \qquad y_0 = y_N = 0$$

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \ddots & & \vdots \\ 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}$$

$$Ay \approx -y''$$

$$y^T A y \approx -\int_0^1 y y'' = -yy'\Big|_0^1 + \int_0^1 (y')^2$$

$$x^T A x \geq 0$$

$$y^T A y = -\sum_{i=1}^{N-1} y_i (y_{i-1} - 2y_i + y_{i+1}) = 2 \sum_{i=1}^{N-1} y_i^2 - 2 \sum_{i=1}^{N} y_i y_{i-1}$$

$$= \sum_{i=1}^{N} (y_i^2 - 2 y_i y_{i-1} + y_{i-1}^2) = \sum_{i=1}^{N} (y_i - y_{i-1})^2 \qquad \begin{bmatrix} y_1 \\ \vdots \\ y_{N-1} \end{bmatrix}$$

then A is symmetric positive definite with bandwidth 1.

<u>Iterative Methods</u>

$A = D + C$

$Ax = b \iff Dx = (b - Cx) \iff x = D^{-1}(b - Cx)$

<u>Method</u>: $x^{(k+1)} = D^{-1}(b - C x^{(k)})$ $\qquad \boxed{D x^{(k+1)} = b - C x^{(k)}}$

Let x^* be the solution of $Ax = b$

Will $x^{(k)} \longrightarrow x^*$?

Obs: $x^* = D^{-1}(b - Cx^*)$

Let $e^{(k)} = x^* - x^{(k)}$ error

$$e^{(k+1)} = x^* - x^{(k+1)} = D^{-1}(b - Cx^*) - D^{-1}(b - Cx^{(k)}) =$$

$$= -D^{-1}C(x^* - x^{(k)}) = -D^{-1}Ce^{(k)}$$

Obs: $C = A - D \qquad -D^{-1}C = -D^{-1}(A - D) = I - D^{-1}A$

$$\boxed{e^{(k+1)} = (I - D^{-1}A)e^{(k)}}$$

Obs: $\|e^{(k+1)}\| \leq \|I - D^{-1}A\| \, \|e^{(k)}\|$

Thus, if $\|I - D^{-1}A\| < 1 \implies e^{(k)} \to 0$

Obs: To compute $x^{(k+1)}$, first compute $d^{(k)} = b - Cx^{(k)}$

then solve $D\, x^{(k+1)} = d^{(k)}$

Obs: this approach makes sense if solving $Dx = \alpha$ is easy