Note Title 2/4/2015

Direct methods to solve linear systems

$A \in \mathbb{R}^{n\times n}$ A nonsingular

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad a_{ij} \in \mathbb{R}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n \qquad A \text{ and } b \text{ given}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$ unknown goal: Find x

$Ax = b$

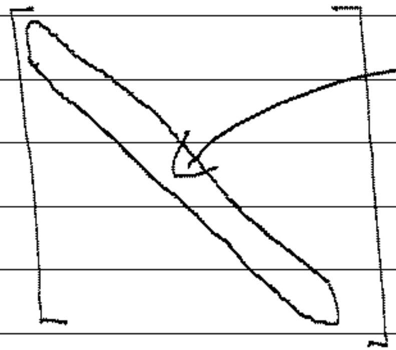
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

<u>Def</u>: $A \in \mathbb{R}^{n\times n}$ is upper triangular if A is of the form

$$A = \begin{bmatrix} x & \cdots & x \\ 0 & \ddots & \vdots \end{bmatrix}$$, ie $a_{ij} = 0$ if $i > j$, all the entries

$$\begin{bmatrix} \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x \end{bmatrix}$$

of A below the diagonal are equal to zero

$A =$  diagonal of A (entries a_{ii} $1 \le i \le n$)

A is lower triangular if $A = \begin{bmatrix} x & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ x & \cdots & \cdots & x \end{bmatrix}$

A is diagonal if $A = \begin{bmatrix} x & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & x \end{bmatrix}$

Solving $Ax=b$ by back substitution if A is upper triangular

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\ddots \qquad \vdots \qquad \vdots$$
$$a_{n-1\,n-1}x_{n-1} + a_{n-1\,n}x_n = b_{n-1}$$
$$a_{nn}x_n = b_n$$

$$x_n = b_n / a_{nn}$$

$$x_k = \left(b_k - \sum_{j=k+1}^{n} a_{kj}x_j \right) / a_{kk}$$ start with $k=n-1$ and

go all the way down to $k=1$

This works when $a_{kk} \neq 0$ for all $1 \le k \le n$, which is the case if and only if A is invertible

algorithm (back substitution)

for $k = n, 1$

  $x_k = b_k$

  for $j = k+1, n$

    $x_k = x_k - a_{kj} x_j$ $\leftarrow 1$

  end

$n-k$

Complexity $\boxed{\dfrac{n(n+1)}{2}}$

$$\sum_{k=1}^{n} n-k+1 = n(n+1)/2$$

$x_k = x_k / a_{kk}$ $\}$ 1

end

<u>Elementary operations</u>

1) Multiply an equation by a non-zero number

2) Replace equation k by equation $k + \alpha$ equation j, where α is any number

3) Exchange two equations

<u>Obs</u>: The solutions of $Ax=b$ does not change if we perform elementary operations

Gaussian Elimination: Perform elementary operations on $Ax=b$ to be left with an upper triangular system

$Ax=b \qquad A^{(1)}=A \qquad b^{(1)}=b$

$$A^{(k)} = \begin{bmatrix} a_{11}^{(k)} & - & - & - & - & a_{1n}^{(k)} \\ 0 & \ddots & & & & \vdots \\ \vdots & \ddots & 0 & a_{kk}^{(k)} & \cdots & a_{kn}^{(k)} \\ \vdots & & \vdots & \vdots & & \\ 0 & \cdots & 0 & a_{nk}^{(k)} & \cdots & a_{nn}^{(k)} \end{bmatrix}$$

1) Find j such that $k \le j \le n$ and $|a_{jk}^{(k)}| \ge |a_{lk}^{(k)}|$ for all $k \le l \le n$

2) Swap equations k and j. this is called pivoting and keeps the algorithm stable

3) Replace equation l by equation $l - \dfrac{a_{lk}^{(k)}}{a_{kk}^{(k)}}$ equation k for $k+1 \le l \le n$

Example: 1) $\begin{bmatrix} \boxed{\begin{matrix} 1 \\ 2 \\ -1 \end{matrix}} & \begin{matrix} 2 & 1 \\ 2 & 3 \\ -3 & 0 \end{matrix} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

$$\begin{array}{c} \cancel{Eq}\,2 \\ \cancel{Eq}\,1 \\ \\ \end{array} \left[\begin{array}{ccc:c} 2 & 2 & 3 & 3 \\ 1 & 2 & 1 & 0 \\ -1 & -3 & 0 & 2 \end{array}\right] \qquad \begin{array}{c} \\ Eq2 - \frac{1}{2} Eq1 \\ Eq3 + \frac{1}{2} Eq1 \end{array} \left[\begin{array}{ccc:c} 2 & 2 & 3 & 3 \\ 0 & \boxed{\begin{matrix} 1 \\ -2 \end{matrix}} & \begin{matrix} -1/2 \\ 3/2 \end{matrix} & \begin{matrix} -3/2 \\ 7/2 \end{matrix} \\ 0 & & & \end{array}\right]$$

$$\begin{array}{l} \\ Eq3 \\ Eq2 \end{array}\left[\begin{array}{ccc:c} 2 & 2 & 3 & 3 \\ 0 & -2 & 3/2 & 7/2 \\ 0 & 1 & -1/2 & -3/2 \end{array}\right] \qquad \left[\begin{array}{ccc:c} 2 & 2 & 3 & 3 \\ 0 & -2 & 3/2 & 7/2 \\ 0 & 0 & 1/4 & 1/4 \end{array}\right] Eq3 + \frac{1}{2} Eq2$$

<u>Algorithm (Gaussian Elimination)</u>

Complexity

for $i = 1 : n-1$

  $k = i$

  for $l = i+1 : n$

    if $|a_{li}| > |a_{ki}|$ then

      $k = l$

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        endif
    end

    for j=i:n
        x = a_ij
        a_ij = a_kj
        a_kj = x
    end

        x = b_i
        b_i = b_k
        b_k = x
```

for $l = i+1 : n$

$x = a_{li} / a_{ii}$ $\quad \}\, 1$

$a_{li} = 0$

for $j = i+1 : n$

$a_{lj} = a_{lj} - x\, a_{ij}$ $\quad \}\, 1 \quad \}\; n-i \quad \}\; n-i+2 \quad \}\; (n-i+2)(n-i)$

end

$b_l = b_l - x\, b_i$ $\quad \}\, 1$

end

end

Complexity $\displaystyle\sum_{i=1}^{n-1} (n-i+2)(n-i) \approx \frac{n^3}{3} = O(n^3)$

Use algorithm to solve $Ax=b$ when A is upper triangular

<u>LU factorization</u> A little extra work lead to finding a permutation P and lower and upper triangular matrices L and U such that $PA = LU$

<u>Cholesky factorization</u>

<u>Def</u>: A^T is the transpose of A

<u>Def</u>: A is symmetric $A = A^T$

Def: $A \in \mathbb{R}^{n\times n}$ $\quad x \in \mathbb{R}^n$ $\quad x^T A x \in \mathbb{R}$

A is positive definite if $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$

and $x^T A x = 0$ only if $x = 0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

Fact: A is symmetric positive definite. then there exists

U upper triangular such that $A = U^T U$

This is called the Cholesky factorization

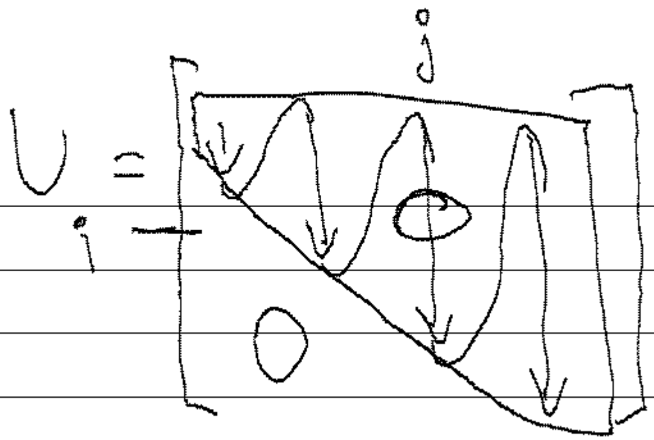
Obs. $A = U^T U$ with U upper triangular. then

$$a_{ij} = \sum_{k=1}^{n} [U^T]_{ik} [U]_{kj} = \sum_{k=1}^{n} u_{ki} u_{kj}$$

Assume

$j \geq i$

$$a_{ij} = \sum_{k=1}^{i} u_{ki} u_{kj}$$



If $i < j$ then

$$a_{ij} = u_{1i} u_{1j} + u_{2i} u_{2j} + \dots + u_{ii} u_{ij}$$

$$u_{ij} = \left(a_{ij} - \sum_{k=1}^{i-1} u_{ki} u_{kj} \right) / u_{ii}$$

If $i = j$

$$a_{ij} = a_{ii} = u_{1i}^2 + u_{2i}^2 + \dots + u_{ii}^2$$

$$u_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2}$$

Algorithm Cholesky   Complexity

for $j=1:n$

  for $i=1:j-1$

    $u_{ij}=a_{ij}$

    for $k=1:i-1$

      $u_{ij}=u_{ij}-u_{ki}u_{kj}$ $\} \; 1$ $\} \; i-1$

    end

    $u_{ij}=u_{ij}/u_{ii}$ $1 \}$ $\} \; i$ $\} \; \frac{j(j-1)}{2}$

  end

  $u_{jj}=a_{jj}$

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for  k=1:j-1
      u_jj = u_jj - u_kj^2          } 1  } j-1  } j
  end
  u_jj = sqrt(u_jj)                 } 1
end
```

Complexity $\sum_{j=1}^{n} \frac{j(j+1)}{2} \approx \frac{n^3}{6} = O(n^3)$