Note Title 2/11/2015

Iterative methods to solve $Ax = b$ $\qquad A = D + C$

$x^{(0)}$ given

for $k = 0, \ldots\ldots$

$d^{(k)} = b - C x^{(k)}$

$x^{(k+1)}$ solution of $D x^{(k+1)} = d^{(k)}$

Example: 1) Jacobi D = diagonal of A $\qquad D = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}$

$$I - D^{-1}A = I - \begin{bmatrix} 1/a_{11} & & 0 \\ & \ddots & \\ 0 & & 1/a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} - \begin{bmatrix} a_{11}/a_{11} & a_{12}/a_{11} & \cdots & a_{1n}/a_{11} \\ a_{21}/a_{22} & a_{22}/a_{22} & \cdots & a_{2n}/a_{22} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}/a_{nn} & a_{n2}/a_{nn} & \cdots & a_{nn}/a_{nn} \end{bmatrix} =$$

$$= - \begin{bmatrix} 0 & a_{12}/a_{11} & \cdots & a_{1n}/a_{11} \\ a_{21}/a_{22} & \ddots & & a_{2n}/a_{22} \\ & & \ddots & \\ a_{n1}/a_{nn} & & & 0 \end{bmatrix}$$

$$\|I - D^{-1}A\|_\infty = \left(\max_{1 \le i \le n} \sum_{j=1}^{n} \left| \frac{a_{ij}}{a_{ii}} \right| \right) - 1$$

Def: A is diagonally dominant if $\sum_{j \neq i} |a_{ij}| < |a_{ii}|$ for all $1 \leq i \leq n$

Example $\begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix}$ is diagonally dominant

Ex $\begin{bmatrix} 4 & -3 & 0 & 0 \\ -1 & 4 & 2 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix}$ is diagonally dominant

Obs: A diagonally dominant $\Rightarrow \|I - D^{-1}A\|_\infty < 1 \Rightarrow$

Jacobi converges

Jacobi iteration

$$d_i^{(k)} = b_i - \sum_{j \neq i} a_{ij} \; x_j^{(k)}$$

$$D \, x^{(k+1)} = d^{(k)}$$

$$x_i^{(k+1)} = d_i^{(k)} / a_{ii} = \left(b_i - \sum_{j \neq i} a_{ij} \; x_j^{(k)} \right) / a_{ii}$$

Algorithm (Jacobi, one iteration)

Input = x (which is $x^{(k)}$)

Output = y (which is $x^{(k+1)}$)

```
for i=1:n
    y_i = b_i
    for j=1:i-1
        y_i = y_i - a_ij x_j          i-1
    end
    for j=i+1:n                           n        n^2
        y_i = y_i - a_ij x_j          n-i
    end
    y_i = y_i / a_ii                  1
end
```

One Jacobi iteration costs n^2

When to stop? $\dfrac{\| x^{(k+1)} - x^{(k)} \|}{\| x^{(k)} \|} < \varepsilon$ for some ε small

<u>Gauss-Seidel</u>

$$D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

for $i = 1:n$

$$x_i^{(k+1)} = \frac{b_i - \sum_{j<i} a_{ij} x_j^{(k+1)} - \sum_{j>i} a_{ij} x_j^{(k)}}{a_{ii}}$$

end

It also converges when A is diagonally dominant

<u>Polynomials</u>: These are function of the form

$$P(x) = a_n x^n + \cdots\cdots + a_1 x + a_0$$

If $a_n \neq 0$, we say that the degree of P is n

<u>Evaluating polynomials</u>

Straight forward

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

# of multiplications	n	$n-1$	1	$= \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$

Better If $n=3$

$$a_3 x^3 + a_2 x^2 + a_1 x + a_0 = \left(\left(\left(a_3 x + a_2 \right) x + a_1 \right) x + a_0 \right)$$

In general

Algorithm

Input: $a_0, a_1, \ldots, a_n$ and x

Output: $p = a_n x^n + \cdots + a_0$

$p = a_n$

for $i = n-1 : 0$

$p = px + a_i$

end

<u>Interpolating a function with a polynomial</u>

Given: (x_i, y_i) $\quad 0 \le i \le n$ $\quad x_i \ne x_j$ if $i \ne j$

Sometimes $f(x)$ is given and $y_i = f(x_i)$

Goal: Find P, polynomial of degree n, or less, such that

$$P(x_i) = y_i \quad \text{for} \quad 0 \le i \le n$$

Lagrange polynomials:

$$L_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \left(\frac{x - x_j}{x_i - x_j} \right)$$

degree of $L_i = n$ $\qquad L_i(x_k) = \begin{cases} 0 & \text{if } k \ne i \\ 1 & \text{if } k = i \end{cases}$

Obs: $P(x) = \sum_{i=0}^{n} y_i L_i(x)$ then $P(x_k) = y_k$

Obs: 1) Evaluating $L_i(x)$ costs $2n$

2) To evaluate $P(x)$ costs $2n(n+1) + (n+1)$. Not efficient

Write P in the form

$$P = c_0 + c_1(x-x_0) + c_2(x-x_1)(x-x_0) + \cdots + c_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

P has degree n. How to find $c_0, \ldots, c_n$ so that $P(x_i) = y_i$ $0 \le i \le n$

Def: $y[x_i] = y_i$

For $i < j$

$$y[x_i, \dots, x_j] = \frac{y[x_{i+1}, \dots, x_j] - y[x_i, \dots, x_{j-1}]}{x_j - x_i}$$

<u>Claim</u> $c_i = y[x_0, \dots, x_i]$

(*) $P(x) = y[x_0] + y[x_0, x_1](x - x_0) + y[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots\dots$

$\dots\dots + y[x_0, x_1, \dots, x_n](x - x_0)(x - x_1)\dots(x - x_{n-1})$ $\quad P(x_i) = y_i$

<u>Example</u>: $(0,2)$ $(1,0)$ $(2,2)$

$y[0] = 2$

$y[1] = 0$

$y[0,1] = \frac{y[1] - y[0]}{1 - 0} = -2$

$y[0,1,2] =$

$$y[2] = 2 \qquad y[1,2] = \frac{y[2]-y[1]}{2-1} = 2$$

$$= \frac{y[1,2]-y[0,1]}{2-0} = 2$$

$$P = 2 - 2(x-0) + 2(x-0)(x-1) \stackrel{?}{=} 2(x-1)^2$$

$$2 - 2x + 2x^2 - 2x \qquad \text{yes} \checkmark$$
$$2(x-1)^2$$

proof that $P(x)$, as in $\circledast$, satisfies $P(x_i) = y_i$

By induction

$n=0$ $\quad P(x) = y[x_0] = y_0 \quad \checkmark \qquad (x_0, y_0)$

$n>0$. Let $r(x) = y[x_0] + \cdots + y[x_0, \ldots, x_{n-1}](x-x_0)\cdots(x-x_{n-2})$

Let $q(x) = y[x_1] + \ldots + y[x_1, \ldots, x_n](x-x_1)\cdots(x-x_{n-1})$

by induction $\quad r(x_i) = y_i \qquad 0 \le i \le n-1$

$$q(x_j) = y_j \qquad 1 \le j \le n$$

Claim $P(x) = \dfrac{(x_n - x)\, r(x) + (x - x_0)\, q(x)}{x_n - x_0}$

Check $P(x_0) = \dfrac{(x_n - x_0)\, r(x_0) + 0}{(x_n - x_0)} = r(x_0) = y_0$

$1 \le i < n \quad P(x_i) = \dfrac{(x_n - x_i)\, r(x_i) + (x_i - x_0)\, q(x_i)}{(x_n - x_0)} = y_i$

$$P(x_n) = 0 + (x_n - x_0)\, q(x_n) / (x_n - x_0) = y_n \quad \checkmark$$

$P(x) = r(x) + \gamma'(x)$ $\qquad P(x_i) = y_i = r(x_i) \qquad 0 \le i \le n-1$

then $\gamma'(x) = \beta(x - x_0)(x - x_1)\cdots\cdots(x - x_{n-1})$

β is the leading coefficient of $P(x)$

$$\beta = \frac{\text{leading coeff of } q - \text{leading coeff of } r}{x_n - x_0}$$

$$\beta = \frac{y[x_1, \ldots, x_n] - y[x_0, \ldots, x_{n-1}]}{x_n - x_0}$$

this completes the proof