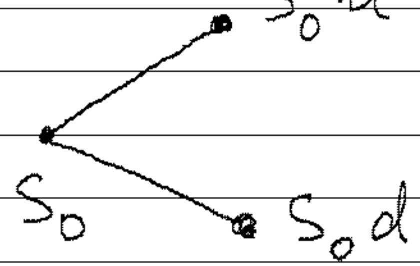
Calibration Binomial Method

Parameters in the Binomial method

u, d, p

$d < u$



$S_1 = S(\delta t)$

$p = P(S_1 = S_0 u)$

$t=0 \quad t=\delta t$

Select u, d, p to match the expectation and variance of

$$dS = rS\,dt + \sigma S\,dW \qquad \text{G B (Geometric Brownian)}$$

From GB $\quad E[S(\delta t)] = S(0)\, e^{r\delta t} \qquad (1)$

$$\mathrm{Var}[S(\delta t)] = (S(0))^2 e^{2r\delta t}\left(e^{\sigma^2 \delta t} - 1\right) \qquad (2)$$

From Binomial $(S(0) = S_0,\ S(\delta t) = S_1)$

$$E[S_1] = p u S_0 + (1-p) d S_0 \qquad (3)$$

$$\mathrm{Var}[S_1] = E[S_1^2] - (E[S_1])^2 = p(S_0 u)^2 + (1-p)(S_0 d)^2 - (S_0 e^{r\delta t})^2 \qquad (4)$$

because $(1) = (3)$

Eqs (2) and (4) imply

$$e^{2r\delta t}\left(e^{\sigma^2 \delta t} - 1\right) = p u^2 + (1-p) d^2 - e^{2r\delta t} \qquad (5)$$

Eqs (1) and (3) imply $\quad p u + (1-p) d = e^{r\delta t} \qquad (6)$

Eq (6) implies $\boxed{p = \dfrac{e^{r\delta t} - d}{u-d}}$ (7)

(Note: p is the risk free probability)

Plug expression of p into

$$pu^2+(1-p)d^2 = \frac{(e^{r\delta t}-d)\,u^2 + (u - e^{r\delta t})\,d^2}{u-d} = \left(\frac{u^2-d^2}{u-d}\right) e^{r\delta t} - ud$$

We set $ud=1$ (for convenience, we have 3 parameters and 2 equations so we are free to impose a third equation).

$$pu^2+(1-p)d^2 = (u+d)\,e^{r\delta t} - 1$$

Back to Equation (5)

$$e^{2r\delta t}\,(e^{\sigma^2\delta t} - 1) = (u+d)\,e^{r\delta t} - 1 - e^{2r\delta t}$$

use $d = 1/u$ to get

$$e^{(2r+\sigma^2)\delta t} = \left(u + \frac{1}{u}\right) e^{r\delta t} - 1$$

Multiply by u, distribute, move all to the right to get

$$0 = u^2 e^{r\delta t} - \left(1 + e^{(2r+\sigma^2)\delta t}\right) u + e^{r\delta t}$$

$$u = \frac{\left(1+e^{(2r+\sigma^2)\delta t}\right) + \sqrt{\left(1+e^{(2r+\sigma^2)\delta t}\right)^2 - 4e^{2r\delta t}}}{2e^{r\delta t}} \qquad (8)$$

δt is small, so inside the square root neglect $O(\delta t^2)$

$$\left(1+e^{(2r+\sigma^2)\delta t}\right)^2 - 4e^{2r\delta t} \cong \left(1 + 1 + (2r+\sigma^2)\delta t\right)^2 - 4\left(1 + 2r\delta t\right) \cong$$

$$\cong 4 + 4(2r+\sigma^2)\delta t - 4 - 8r\delta t = 4\sigma^2\delta t$$

thus $\sqrt{(1+e^{(2r+\sigma^2)\delta t})^2 - 4e^{2r\delta t}} = \sqrt{4\sigma^2\delta t + O(\delta t^2)} = 2\sigma\sqrt{\delta t} + O(\delta t^{3/2})$

because $\sqrt{4\sigma^2\delta t + O(\delta t^2)} = 2\sigma\sqrt{\delta t}\,\sqrt{1 + \frac{O(\delta t^2)}{4\sigma^2\delta t}} = 2\sigma\sqrt{\delta t}\,\sqrt{1+O(\delta t)} =$

$= 2\sigma\sqrt{\delta t}\left(1+O(\delta t)\right) = 2\sigma\sqrt{\delta t} + O(\delta t^{3/2})$

Plug this last expression into Equation (8) to get

$$u = \frac{\left(1+e^{(2r+\sigma^2)\delta t}\right) + 2\sigma\sqrt{\delta t} + O(\delta t^{3/2})}{2e^{r\delta t}}$$

$$u = \left[\frac{\left(1+1+(2r+\sigma^2)\delta t + 2\sigma\sqrt{\delta t}\right) + O(\delta t^{3/2})}{2}\right] e^{-r\delta t}$$

$$u = \left[1 + \left(r + \frac{\sigma^2}{2}\right)\delta t + \sigma\sqrt{\delta t} + O(\delta t^{3/2})\right]\left[1 - r\delta t + O(\delta t^2)\right]$$

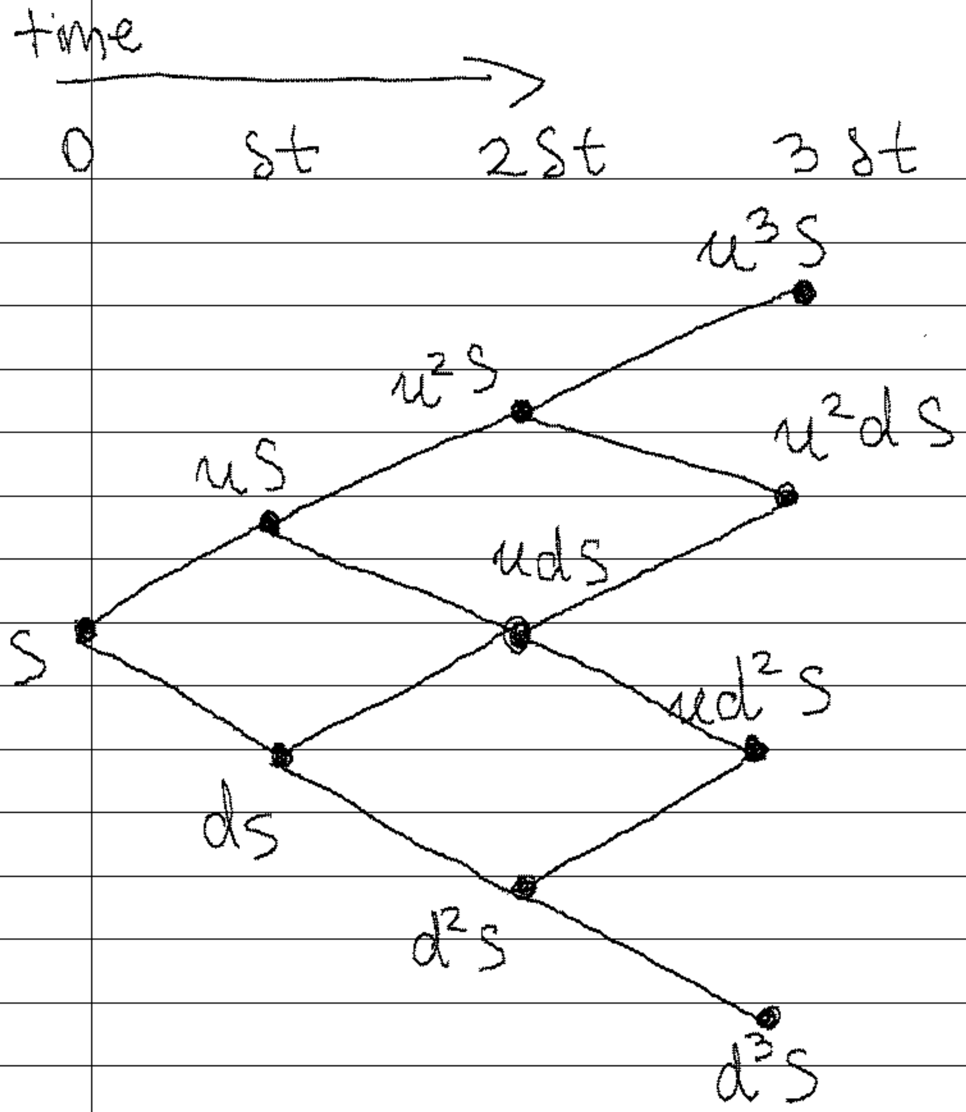
$$u = 1 - r\delta t + \left(r + \frac{\sigma^2}{2}\right)\delta t + \sigma\sqrt{\delta t} + O(\delta t^{3/2})$$

$$u = 1 + \sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2\delta t + O(\delta t^{3/2}) = e^{\sigma\sqrt{\delta t}} + O(\delta t^{3/2})$$

$u = e^{\sigma\sqrt{\delta t}}$	$d = e^{-\sigma\sqrt{\delta t}}$	$p = \dfrac{e^{r\delta t} - d}{u - d}$

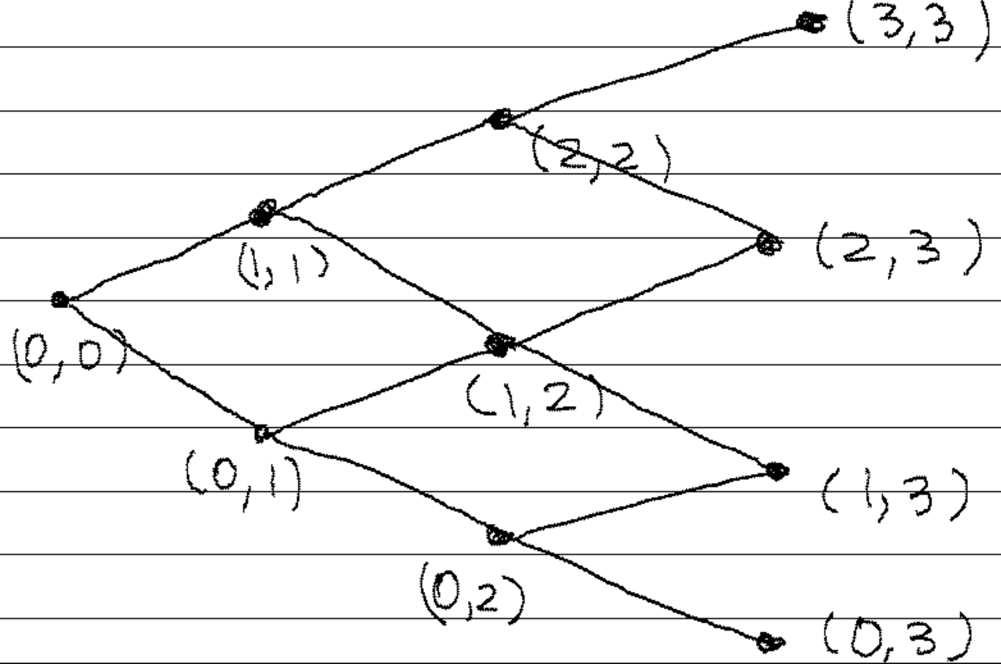
<u>Several time steps. Binomial</u>

T = expiry $\qquad S = S(0) \qquad \delta t = \dfrac{T}{N}$



We call this graph the binomial tree or lattice

<u>Labeling the nodes</u>



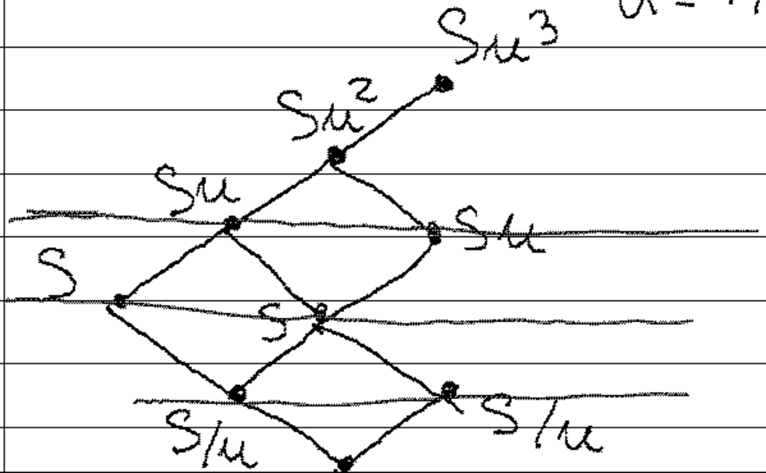
(i,j) j means at time $j\,\delta t$

i means the asset price went up i times, i.e. the asset price is $S u^i d^{j-i}$

S_{ij} = value of the asset at the node (i,j)

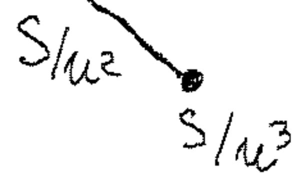
$$S_{ij} = S u^i d^{j-i} = S u^{2i-j}$$

$d = 1/u$



$$S_{ij} = S_{i-1\,j-2}$$

$$S_{ij} = S u^{2i-j} \quad S_{i-1\,j-2} = S u^{2(i-1)-(j-2)}$$



Algorithm (Computation of S_{ij})

Input: S, u, N

Output: S_{ij}

$S_{00} = S$

$S_{01} = S_{00}/u$

$S_{11} = S_{00}\,u$

for $j = 2, N$

    $S_{0j} = S_{0\,j-1}/u$

    for $i = 1, j-1$

$$S_{ij} = S_{i-1\, j-2}$$

end

$$S_{jj} = S_{j-1\, j-1}\, u$$

end

f_{ij} = value of the option at the node (i, j)

Once we have S_{iN} $(0 \le i \le N)$, the asset values at time T, we can compute f_{iN}.

Algorithm (Computation of f_{iN})

Input: S_{iN} $(0 \le i \le N)$, K Output: f_{iN}

If option is call, then

for $i = 0, N$

$f_{iN} = \max\{S_{iN} - K, 0\}$

end

else if option is put, then

for $i = 0, N$

$f_{iN} = \max\{K - S_{iN}, 0\}$

end

end

Once we have f_{iN} for all i, we go back in time to compute

f_{ij} $\quad j<N$ and $0 \le i \le j$

Algorithm (Computation of f_{ij})

Input: f_{iN}, p, $\alpha = e^{-r\delta t}$, $q = 1-p$

Output: f_{ij}

$x = \alpha p$

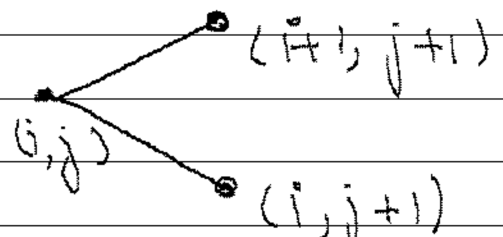
$y = \alpha q$

for $j = N-1, 0$

  for $i = 0, j$

    $f_{ij} = x f_{i+1\, j+1} + y f_{i\, j+1}$

<u>Recall</u>: $f_{ij} = e^{-r\delta t}\left(p f_{i+1\, j+1} + (1-p) f_{i\, j+1}\right)$



Let $x = e^{-r\delta t} p$ and

$y = e^{-r\delta t}(1-p)$

then $f_{ij} = x f_{i+1\, j+1} + y f_{i\, j+1}$

end
end

To finally get $f = f_{00}$, the value of the option presently

Putting all together

Algorithm (Binomial)

Input: $T, N, K, r, \sigma, S\ (=S(0))$

Output: f_{ij}, S_{ij}

$\delta t = T/N$

$u = e^{\sigma\sqrt{\delta t}}$

Algorithm to compute S_{ij}

Algorithm to compute f_{iN}

$\alpha = e^{-r\delta t}$

$d = 1/u$

$$p = \frac{1/\alpha - d}{u - d}$$

$q = 1 - p$

Algorithm to compute f_{ij}

Present value of the option $= f_{00}$