Note Title 1/21/2015

Black-Scholes model

$dS = \mu S\, dt + \sigma S\, dW$

$f(S,t)$ = value of the option at time t when the value of the asset at that time is S

Portfolio: Short in an option + long in A shares of stock

Value of portfolio = $g = AS - f(S,t)$

Reminder of Ito's lemma

$dX = a\, dt + b\, dW$ and $F(X,t)$ then

$$dF = \left(a\frac{\partial F}{\partial X} + \frac{\partial F}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 F}{\partial X^2}\right) dt + b\frac{\partial F}{\partial X}\, dW$$

In our case $X = S$, $a = \mu S$, $b = \sigma S$ and $F = g$

$$dg = \left(\mu S \frac{\partial g}{\partial S} + \frac{\partial g}{\partial t} + \frac{1}{2}(\sigma S)^2 \frac{\partial^2 g}{\partial S^2}\right) dt + \sigma S \frac{\partial g}{\partial S} dW$$

$$dg = \left[\mu S \left(A - \frac{\partial f}{\partial S}\right) - \frac{\partial f}{\partial t} + \frac{1}{2}(\sigma S)^2 \left(-\frac{\partial^2 f}{\partial S^2}\right)\right] dt + \sigma S \left[A - \frac{\partial f}{\partial S}\right] dW$$

Set $A = \frac{\partial f}{\partial S}(S, t)$ in $[t, t+dt)$

$$dg = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt$$

So in dt g increases by $\left(-\frac{\partial f}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt$ no randomness

thus, no arbitrage implies that this should be equal to

$$r\underline{g}\,dt = r\left(S\frac{\partial f}{\partial S} - f\right)dt$$

then

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0$$

Black-Scholes equations

Assumptions used to get Black-Sholes

1) No arbitrage

2) Frictionless market (no transaction costs, borrowing and lending interest rates are the same, etc....)

3) Asset price follows a geometric Brownian motion

4) r & σ are constants for $0 \le t \le T$. No dividends are paid in this period of time. The option is European

Need to impose terminal condions to solve Black-Scholes, ie we need to prescribe $f(S,T)$, ie f at $t=T$. This is the payoff function

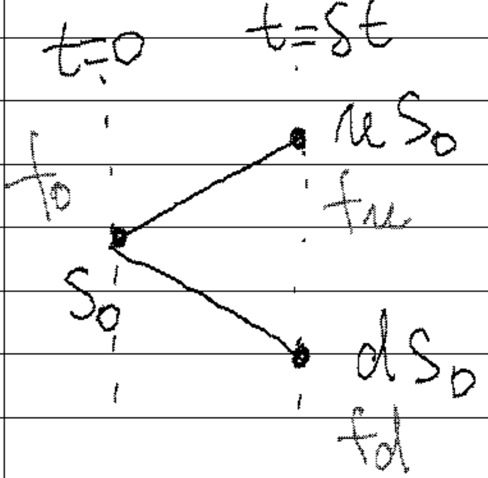
For calls $f(S,T) = \max\{S-K, 0\}$

For puts $f(S,T) = \max\{K-S, 0\}$

Binomial

$S_0 =$ asset price at $t=0$

$S_1 =$ asset price at $t=\delta t$



$p(S_1 = uS_0) = p_u$ $\quad u > d$

$p(S_1 = dS_0) = p_d$

$p_u + p_d = 1$ $\quad p_u > 0$ & $p_d > 0$

European option

Exercise time $= \delta t$

$f_0 =$ value of the option at $t=0$

Goal: find f_0

f_u = value of option at $t=\delta t$ if $S_1 = u S_0$

f_d = value of option at $t=\delta t$ if $S_1 = d S_0$

Obs: 1) We know f_u & f_d

If it is a call option $f_u = \max\{u S_0 - K, 0\}$

$f_d = \max\{d S_0 - K, 0\}$

If it is a put option $f_u = \max\{K - u S_0, 0\}$

$f_d = \max\{K - d S_0, 0\}$

2) No arbitrage $\Longrightarrow$ $d < e^{r\delta t} < u$

Portfolio P_1 : an option

Portfolio P_2 : α shares of the asset + β cash

at $t = \delta t$

S_1	Value of P_2	Value of P_1
$S_0 u$	$\alpha S_0 u + \beta e^{r\delta t}$	f_u
$S_0 d$	$\alpha S_0 d + \beta e^{r\delta t}$	f_d

Select α & β so value of P_1 = value of P_2 at $t = \delta t$ regardless of the value of S_1

$$\alpha S_0 u + \beta e^{r\delta t} = f_u$$
$$\alpha S_0 d + \beta e^{r\delta t} = f_d$$

then

$$\alpha = \frac{f_u - f_d}{(u-d) S_0} \qquad \beta = \frac{(d f_u - u f_d)}{d-u} e^{-r\delta t}$$

Since value of P_1 = value of P_2 at $t = \delta t$, no arbitrage implies value of P_1 = value of P_2 at $t = 0$

Value of P_2 at $t=0$ $= \alpha S_0 + \beta$

Value of P_1 at $t=0$ $= f_0$

Then $f_0 = \alpha S_0 + \beta = \dfrac{f_u - f_d}{u-d} + \dfrac{u f_d - d f_u)}{u-d} e^{-r\delta t}$

$$f_0 = e^{-r\delta t} \left\{ \left(\frac{e^{r\delta t} - d}{u-d}\right) f_u + \left(\frac{u - e^{r\delta t}}{u-d}\right) f_d \right\}$$

<u>Def</u>: $\boxed{\pi_u = \dfrac{e^{r\delta t} - d}{u-d}}$ $\boxed{\pi_d = \dfrac{u - e^{r\delta t}}{u-d}}$

<u>Obs</u>: 1) $\pi_u + \pi_d = 1$

2) $\pi_u > 0$ & $\pi_d > 0$. So we can interpret them as probabilities

<u>Def</u>: f_1 = value of the option at $t = \delta t$

<u>Obs</u> $P(f_1 = f_u) = P(S_1 = u S_0) = p_u$. Note that f_0 is independent of p_u

$E(f_1) = p_u f_u + p_d f_d$

<u>Def</u>: $\hat{E}(f_1) = \pi_u f_u + \pi_d f_d$ is the expected value with respect to this new probability π_u and π_d.

<u>Notation</u>: this «artificial probability» π_u & π_d is called risk-neutral

<u>Obs</u> $f_0 = e^{-r\delta t}\, \hat{E}(f_1)$

<u>Obs</u> $\begin{matrix} p_u = \pi_u \\ p_d = \pi_d \end{matrix} \iff E(S_1) = \hat{E}(S_1)$

Algorithm (One step binomial)

Input: $S_0, u, d, K, r, \delta t$

Output: f_0

If call then

$f_u = \max\{S_0 u - K, 0\}$

$f_d = \max\{S_0 d - K, 0\}$

If put then

$f_u = \max\{K - S_0 u, 0\}$

$f_d = \max\{K - S_0 d, 0\}$

$\pi_u = \dfrac{e^{r\delta t} - d}{u - d}$

$\pi_d = 1 - \pi_u$

$$f_0 = e^{-r\delta t}\left(\pi_u f_u + \pi_d f_d\right)$$

Calibration: How do we pick u, d and p?

We assume the asset value follows a geometric Brownian motion

$$dS = \mu S\, dt + \sigma S\, dW$$

we are interested in the value of the option, which does not depend on μ. If we change μ, the value of the option does not change. Note

$$E(S) = S(0)\, e^{\mu t}$$

After δt, $E(S(\delta t)) = S(0)\, e^{\mu \delta t}$

If we change μ by r, we will get the risk-neutral probabilities. This is a widely adopted strategy that we will follow

$$dS = rS\,dt + \sigma S\,dW$$

<u>Obs</u> $\log(S(\delta t)) \sim \mathcal{N}\left(\log(S(0)) + \left(r - \frac{\sigma^2}{2}\right)\delta t,\ \sigma^2 \delta t\right)$

Thus, $E[S(\delta t)] = S(0)\, e^{r\delta t}$ (1)

$\mathrm{Var}[S(\delta t)] = (S(0))^2\, e^{2r\delta t}\left(e^{\sigma^2 \delta t} - 1\right)$ (2)

Pick u, d & p_u so the expectation and variance using the binomial are the same as equations (1) and (2)

Set $u = \frac{1}{d}$ (using the freedom we have because we have two equations with 3 unkowns)

From binomial Set $p = p_u$ $S = S_0 = S(0)$

$$E[S(\delta t)] = E[S_1] = pSu + (1-p)Sd \qquad (3)$$

$$Var[S(\delta t)] = E[S_1^2] - (E[S_1])^2 = p(Su)^2 + (1-p)(Sd)^2 - $$
$$- (pSu + (1-p)Sd)^2 \qquad (4)$$