Note Title 1/14/2015

Modeling the dynamics of asset prices

Discrete modeling

$[0,T]$ $\delta t = T/N$

$S_\ell = S$ at time $\ell\,\delta t$

(1) $S_{\ell+1} = u_\ell S_\ell$

(2) u_ℓ are non-negative random variables, S_0 given

(3) u_ℓ are i.i.d.

Set $z_\ell = \log u_\ell$

(4) $\log S_{\ell+1} = \log S_\ell + z_\ell$

(5) $z_\ell \sim \mathcal{N}(\nu \delta t, \sigma^2 \delta t)$

Thus u_ℓ is lognormally distributed

<u>Obs</u>: a) $\log S_\ell = \log S_{\ell-1} + z_{\ell-1} = \log S_{\ell-2} + z_{\ell-2} + z_{\ell-1}$

$$\log S_\ell = \log S_0 + z_0 + z_1 + \cdots + z_{\ell-1}$$

b) $\log S_\ell \sim \mathcal{N}(\log S_0 + \nu \ell \delta t, \sigma^2 \delta t \ell)$

$\ell \delta t = t \qquad S_\ell = S(t)$

$$\log S(t) \sim \mathcal{N}(\log S_0 + \nu t, \sigma^2 t)$$

<u>Continuous time modeling</u>

$dt = \delta t \qquad t = \ell \delta t$

(6) $\log S(t+dt) - \log S(t) = z_\ell$

Set $z_\ell = \nu dt + \sigma(W(t+dt) - W(t))$ (7)

then $\sigma(W(t+dt) - W(t)) = z_\ell - \nu dt \sim \mathcal{N}(0, \sigma^2 dt)$

Def: $X(t)$ is a stochastic process if $X(t)$ is a random variable for each t.

Notation $dX(t) = X(t+dt) - X(t)$

(8) $d \log S(t) = \nu dt + \sigma dW(t)$

Notation: $\int_s^t dX(r) = \sum_{i=0}^{N-1} dX(s+idt) = \sum_{i=0}^{N-1} X(s+(i+1)dt) - X(s+idt) =$

$= X(t) - X(s)$ where $dt = \frac{t-s}{N}$

<u>Obs</u>: $\sigma(W(t)-W(s)) = \sum_{i=0}^{N-1} \sigma\, dW(s+idt) =$

$\sum_{i=0}^{N-1} \sigma(W(s+(i+1)dt) - W(s+idt)) \sim \mathcal{N}(0, \sigma^2 N dt) = \mathcal{N}(0, \sigma^2(t-s))$

<u>Def</u>: $W(t)$ stochastic process is a standard Wiener process if

1) $W(0) = 0$

2) $W(t) - W(s) \sim \mathcal{N}(0, t-s)$

3) $t_1 < t_2 \leq t_3 < t_4$ then $W(t_2) - W(t_1)$ and $W(t_4) - W(t_3)$ are independent

<u>Obs</u>: 3) comes from the z_ℓ being independent

<u>Stochastic integrals</u>

__Def__. $W(t)$ Wiener process. $X(t)$ stochastic process that depends on $W(s)$ for $0 \le s < t$.

Set $t_k = \frac{kT}{N}$

$t_0 = 0 \quad t_1 \quad t_2 \quad \ldots \quad T = t_N$

$$\int_0^T X(t)\,dW(t) = \lim_{N\to\infty} \sum_{k=0}^{N-1} X(t_k)\left[(W(t_{k+1}) - W(t_k))\right]$$

__Obs__: $X(t_k)$ and $W(t_{k+1}) - W(t_k)$ are independent

thus

$$E\left[\int_0^T X(t)\,dW(t)\right] \approx E\left[\sum_{k=0}^{N-1} X(t_k)\left[W(t_{k+1}) - W(t_k)\right]\right] =$$

$$= \sum_{k=0}^{N-1} E\left[X(t_k)\right] \underbrace{E\left[W(t_{k+1}) - W(t_k)\right]}_{=0} = 0$$

Ito's lemma

W = Wiener process

(a) $dX = a(X,t)\,dt + b(X,t)\,dW$

$G = G(x,y)$ a function of two variables

Taylor expansion

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(dx)^2 + \frac{\partial^2 G}{\partial x \partial y}dx\,dy + \frac{1}{2}\frac{\partial^2 G}{\partial y^2}(dy)^2 + \cdots$$

$F = F(X,t)$, where X satisfies (a)

note: $dW = W(t+dt) - W(t) \sim \mathcal{N}(0, dt)$

thus $dW = \phi(t)\sqrt{dt}$ where $\phi(t) \sim \mathcal{N}(0,1)$

Back to F

$$dF = \frac{\partial F}{\partial X} dX + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} (dX)^2 + O((dt)^{3/2})$$

$$dF = \frac{\partial F}{\partial X}(a\,dt + b\,dW) + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} b^2 (dW)^2 + O((dt)^{3/2})$$

$$(dW)^2 \longrightarrow dt$$

<u>Ito's lemma:</u> W Wiener

$$dX = a(X,t)\,dt + b(X,dt)\,dW$$

$F = F(X,t)$, then

$$dF = \left(a\frac{\partial F}{\partial X} + \frac{\partial F}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 F}{\partial X^2} \right) dt + b \frac{\partial F}{\partial X} dW$$

Geometric Brownian motion This is defined by the stochastic differential equation

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t)$$

W Wiener

$$X = S \qquad a = \mu S \qquad b = \sigma S$$

μ & σ are constants μ = drift σ = volatility

Let $F(S,t) = \log S$

$$\frac{\partial F}{\partial S} = \frac{1}{S} \qquad \frac{\partial F}{\partial t} = 0 \qquad \frac{\partial^2 F}{\partial S^2} = -\frac{1}{S^2}$$

$$dF = \left(\mu S \frac{1}{S} - \frac{1}{2}\sigma^2 S^2 \frac{1}{S^2}\right) dt + \sigma S \frac{1}{S}\, dW$$

$$\boxed{d \log S = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW}$$

Integrate

$$\log S(t) = \log S(0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma\, W(t)$$

$$\log S(t) \sim N\left[\log S(0) + \left(\mu - \frac{1}{2}\sigma^2\right) t,\ \sigma^2 t\right]$$

$$S(t) = S(0)\, e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}$$

$S(t)$ is lognormally distributed

$$E[S(t)] = S(0)\, e^{\mu t}$$

Application of the no arbitrage principle

1) Price of a forward contract

A forward contract is an agreement to sell an asset at a specified future date at a predetermined price F.

r = risk free interest rate

<u>Portfolio 1</u> (person selling the asset at time T for F)

a) Borrow $S(0)$ at time 0 to buy the asset
b) Sell for F at time T
c) Pay back the loan, i.e. pay back $S(0)e^{rT}$

If $F > S(0)e^{rT}$, then he made a risk free profit from zero investment. Not allowed by the no arbitrage principle

Thus $F \leq S(0)e^{rT}$

Portfolio 2 (person buying the asset at time T for F)

a) Sell short the asset at time 0 to get $S(0)$

b) Invest that money with the risk free interest to have $S(0)e^{rT}$ at time T

c) Buy the asset at time T for F

If $S(0)e^{rT} - F > 0$, then he made a risk free profit with 0 initial investment. Not allowed by the no arbitrage principle. Thus $F \geq S(0)e^{rT}$

Thus $\boxed{F = S(0)e^{rT}}$

Put-call parity (No arbitrage principle. Example)

C and P are the prices of a call and a put, both European, with the same exercise price K and maturity T.

Portfolio P_1 at $t=0$

one call option + Ke^{-rT} in cash

Portfolio P_2 at $t=0$

one put option + 1 share of the underlying

$\boxed{\text{Value of } P_1 \text{ at } t=0}$	$C + Ke^{-rT}$
$\boxed{\text{Value of } P_2 \text{ at } t=0}$	$P + S(0)$

$$\boxed{\text{Value of } P_1 \text{ at } t=T} = \begin{cases} S(T) - K + Ke^{-rT}e^{rT} & \text{if } S(T) > K \\ Ke^{-rT}e^{rT} & \text{if } S(T) \le K \end{cases}$$

$$\boxed{\text{Value of } P_1 \text{ at } t=T} = \max\{S(T), K\}$$

$$\boxed{\text{Value of } P_2 \text{ at } t=T} = \begin{cases} 0 + S(T) & \text{if } S(T) > K \\ K - S(T) + S(T) & \text{if } S(T) \le K \end{cases}$$

$$\boxed{\text{Value of } P_2 \text{ at } t=T} = \max\{S(T), K\}$$

$$\text{Value of } P_1 \text{ at } T = \text{Value of } P_2 \text{ at } T$$

thus, no arbitrage principle implies

Value of P_1 at time 0 = Value of P_2 at time 0

$$\boxed{C + Ke^{-rT} = P + S(0)}$$