Random variables

$X : S \longrightarrow \mathbb{R}$

Example: Throw two dice

$S = \{ (i,j) : 1 \le i \le 6,\ 1 \le j \le 6,\ i \text{ and } j \text{ integers} \}$

$X(i,j) = i + j$ (the sum of the two numbers you get)

Names: X is called a discrete random variable if it can only take values in a countable set $\{x_1, x_2, \dots, x_n, \dots\}$ (as in the example above)

Otherwise X is called a continuous random variable

Def: Let X be a discrete random variable that can only take the values $x_1, x_2, \dots, x_n, \dots$. The probability mass function $p(.)$ of X is

$$p(x_i) = P(X = x_i)$$

Example two dice. X = sum of both numbers

dice 2 \ dice 1	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

x_i	$p(x_i)$
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36

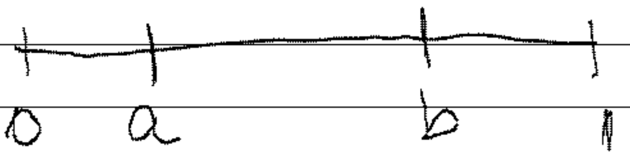
x_i	$p(x_i)$
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

Def. Let X be a continuous random variable. We say that $f(x)$ is the probability density function of X if, for all $a<b$,

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

Example: Pick a real number between 0 and 1 with «uniform» probability distribution, i.e.

$$f(x) = \begin{cases} 1 & \text{if } 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



$$P(a \le X \le b) = \int_a^b f(x)\,dx = \int_a^b dx = b-a$$

$$0 \le a \le b \le 1$$

<u>Def</u>: The expected value of a random variable X is

$$E[X] = \sum_i x_i \, p(x_i) \qquad \text{discrete case}$$

$$E[X] = \int_{-\infty}^{\infty} x \, f(x) \, dx \qquad \text{continuum case}$$

<u>Example</u> 1) X continuum random variable with uniform probability distribution in $[0,1]$

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x \, f(x) dx = \int_0^1 x \, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

2) Flip a coin. $X = \begin{cases} 1 & \text{if heads} \\ 0 & \text{if tails} \end{cases}$

$$E[X] = \frac{1}{2} 1 + \frac{1}{2} 0 = \frac{1}{2}$$

$$p(1) = \frac{1}{2} \qquad p(0) = \frac{1}{2}$$

Def: The variance of a random variable X is

$$\mathrm{Var}(X) = E\left[(X - E[X])^2 \right]$$

Example: Flip a coin $\quad X = \begin{cases} 1 & \text{heads} \\ 0 & \text{tails} \end{cases}$

$E[X] = \frac{1}{2}$ $\quad \boxed{\mathrm{Var}(X) = E\left[\left(X - \frac{1}{2}\right)^2\right] = \left(0 - \frac{1}{2}\right)^2 p(0) + \left(1 - \frac{1}{2}\right)^2 p(1) = \frac{1}{4}}$

Obs: X random variable. $g(X)$ is also a random variable, where g is a real valued function.

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx \qquad \text{continuum case}$$

$$E[g(X)] = \sum_i g(x_i)\, p(x_i) \qquad \text{discrete case}$$

Example: X uniform in $[0,1]$ $\quad E[X] = \frac{1}{2}$

$$\boxed{Var(X) = E\left[\left(X-\frac{1}{2}\right)^2\right] = \int_{-\infty}^{\infty}\left(x-\frac{1}{2}\right)^2 f(x)\,dx = \int_0^1 \left(x-\frac{1}{2}\right)^2 dx = \frac{1}{3}\left(x-\frac{1}{2}\right)^3\Big|_0^1 = \frac{1}{12}}$$

Def: Standard deviation $\sigma = \sqrt{Var(X)}$

Obs: σ small means small spread X will likely take values close to $E[X]$. σ large is the opposite.

Properties $a, b \in \mathbb{R}$

1) $E[aX+b] = aE[X] + b$

2) $Var(X) = E[X^2] - (E[X])^2$

3) $Var(aX+b) = a^2\, Var(X)$

Normal random variable: a continuous random variable with probability density function

(*) $$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Notation: $X \sim \mathcal{N}(\mu, \sigma^2)$ means X is a normal random variable with probability density function (*)

Obs: $X \sim \mathcal{N}(\mu, \sigma^2)$ then

$E[X] = \mu$ and $Var[X] = \sigma^2$

Obs: $\int_0^\infty e^{-x^2}\,dx = I$. then

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right) = \int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dx\,dy =$$

$$= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\, r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{e^{-r^2}}{-2} \Big|_0^\infty \right) d\theta =$$

$$= \int_0^{\pi/2} \frac{1}{2}\,d\theta = \frac{\pi}{4} \qquad I = \frac{\sqrt{\pi}}{2}$$

Def: Z is a unit or standard normal random variable if $Z \sim \mathcal{N}(0,1)$

Property: $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\alpha N + \beta \sim \mathcal{N}(\alpha\mu + \beta, \alpha^2\sigma^2)$.

In particular, $\frac{(X-\mu)}{\sigma} \sim N(0,1)$

Def: Z is a lognormal random variable if $\log Z$ is a normal random variable, i.e. $Z = e^X$ where $X \sim N(\mu, \sigma^2)$

Properties: If $X \sim N(\mu, \sigma^2)$ and $Z = e^X$, then

$$E[Z] = e^{\mu + \frac{\sigma^2}{2}} \quad \text{and} \quad Var[Z] = e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$$

Def: X & Y random variables. They are independent if, for all a and b,

$$P\{X \le a \text{ and } Y \le b\} = P(X \le a)\, P(Y \le b)$$

Obs: X and Y independent. Then

1) If they are both discrete, $p(x,y) = p_X(x)\, p_Y(y)$,
$p(x,y) = P(X=x \text{ and } Y=y)$

2) If they are both continuous, then $f(x,y) = f_X(x) f_Y(y)$
where $f(x,y)$ is the joint probability distribution, i.e.
$$P\{(X,Y) \in A\} = \iint_A f(x,y)\,dx\,dy \quad \text{for any } A \subset \mathbb{R}^2$$

f_X and f_Y are the distributions of X and Y respectively

Obs: X & Y are independent, then $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$

Obs: $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$

X and Y are independent, then

$X+Y \sim \mathcal{N}(\mu_X+\mu_Y, \sigma_X^2+\sigma_Y^2)$

Strong law of large numbers: $X_1, X_2, \ldots, X_n, \ldots$ a sequence of independent random samples from the same distribution with expectation μ, the

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} x_i}{n} = \mu \quad \text{with probability } 1$$

Modeling the dynamics of asset prices

Discrete modeling

$[0, T]$ = time interval

$\delta t = \frac{T}{N}$ $\qquad$ S = price of the asset

$S_l = S$ at time $t = l\,\delta t$ $\qquad$ $l = 0, 1, 2, \ldots, N$

Model: $S_{l+1} = u_l S_l$ $\qquad$ (1)

this means the price increased if $u_l > 1$ by $(u_l - 1)\,100\%$ and decreased if $0 < u_l < 1$ by $(1 - u_l)\,100\%$

(2) u_l is a non-negative random variable, S_0 is given

(3) all u_ℓ are i.i.d, i.e. independent identically distributed

(1), (2) and (3) are modeling assumptions. True??