Def: $A \in \mathbb{R}^{2\times 2}$. $tr(A)$ = trace of A =

$= a_{11} + a_{22}$

Obs: $P(\lambda) = \det(A - \lambda I) =$

$= \lambda^2 - tr(A)\lambda + \det(A) = (\lambda - \lambda_1)(\lambda - \lambda_2)$

λ_1 & λ_2 are the eigenvalues of A.

They may be the same. They may be complex

$tr(A) = \lambda_1 + \lambda_2$ $\det(A) = \lambda_1 \lambda_2$

$\boxed{\text{Case 1}}$ $(tr A)^2 - 4\det(A) > 0$. Two real eigenvalues

$\boxed{\text{Case 1.1}}$ $\det(A) < 0 \Rightarrow \lambda_1 < 0 < \lambda_2$

In this case 0 is a saddle

$\boxed{\text{Case 1.2}}$ $\det(A) > 0$

Case 1.2 $\det(A) > 0$

Case 1.2.1 $\operatorname{tr}(A) > 0 \Rightarrow 0 < \lambda_1 < \lambda_2$

In this case 0 is a source

Case 1.2.2 $\operatorname{tr}(A) < 0 \Rightarrow \lambda_1 < \lambda_2 < 0$

In this case 0 is a sink

Case 2 $(\operatorname{tr}A)^2 - 4\det(A) = 0$ one real eigenvalue

Case 2.1 $\operatorname{tr}(A) > 0 \Rightarrow 0 < \lambda_1$

In this case 0 is a source

Case 2.2 $\operatorname{tr}(A) < 0 \Rightarrow \lambda_1 < 0$

In this case 0 is a sink.

Case 3 $(\operatorname{tr}A)^2 - 4\det(A) < 0$

2 complex eigenvalues $\lambda = \alpha \pm i\beta$.

Note that $tr(A) = 2\alpha = 2\,Re(\lambda)$

$\boxed{\text{Case 3.1}}$ $tr A > 0$. $\Rightarrow Re(\lambda) > 0$

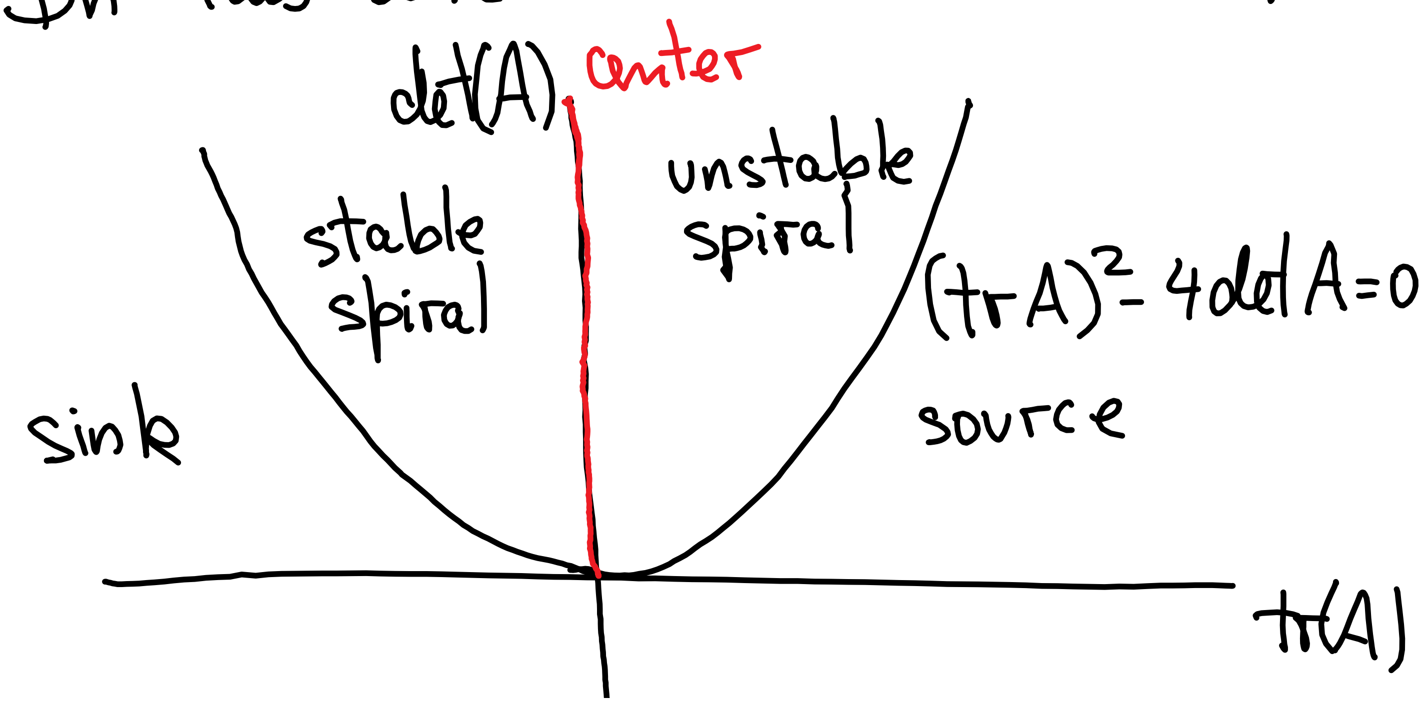
In this case 0 is an unstable spiral

$\boxed{\text{Case 3.2}}$ $tr A = 0 \Rightarrow Re(\lambda) = 0$

In this case 0 is a center

$\boxed{\text{Case 3.3}}$ $tr(A) < 0 \Rightarrow Re(\lambda) < 0$

In this case 0 is a stable spiral.



$\pi(A)$

saddle

$\mathbb{R}^n$ = set of n-vectors $= \left\{ x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R} \right\}$

<u>Def</u>: Let $v_1, v_2, \dots, v_r \in \mathbb{R}^n$. We say that they are linearly dependent if there exists $c_1, c_2, \dots, c_r \in \mathbb{R}$ not all equal to zero such that

$$c_1 v_1 + c_2 v_2 + \dots + c_r v_r = 0$$

They are linearly independent if they are not linearly dependent.

<u>Def</u>: Let $v_1, \dots, v_r \in \mathbb{R}^n$. The span of this set of vectors is

$$\operatorname{span}\{v_1, v_2, \dots, v_r\} =$$

$$= \{ c_1 v_1 + c_2 v_2 + \dots + c_r v_r : \ c_i \in \mathbb{R} \}$$

<u>Def</u>: $S \subset \mathbb{R}^n$. S is a subspace if

1) $0 \in S$
2) $v \in S \Rightarrow \lambda v \in S$
3) $v_1, v_2 \in S \Rightarrow v_1 + v_2 \in S$

<u>Obs</u>: $\mathbb{R}^n$ is a vector space. There are other vector spaces. The notion of a subspace applies to any vector space.

<u>Def</u>: Let S be a subspace. We say that $v_1, \dots, v_r$ spans S if

$$S = \operatorname{span}\{v_1, v_2, \dots, v_r\}$$

Def: Let S be a subspace. $v_1, \dots, v_r$ is a basis for S if $v_1, \dots, v_r$ spans S and $v_1, \dots, v_r$ is linearly independent.

Obs: Let S be a subspace. Let $v_1, \dots, v_r$ be a basis for S. Let $x \in S$. there there exists $\boxed{\text{unique}}$ coefficients $c_1, \dots, c_r$ such that

$$x = c_1 v_1 + \dots + c_r v_r$$

These numbers $c_1, \dots, c_r$ are called the coordinates of x in the basis $v_1, \dots, v_r$

Obs: Let S be a subspace. Let $v_1, \dots, v_r$ be a basis of S. Let $x \in S$.

$$X = c_1 v_1 + c_2 v_2 + \dots + c_r v_r =$$

$$= \underbrace{\begin{bmatrix} v_1 & v_2 & \dots & v_r \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{bmatrix}}_{= c}$$

To find c, solve the linear system

$$Vc = X$$

X is known. V is the matrix whose i^{th} column is v_i. c is the unknown.

<u>Obs</u>: Gaussian elimination to solve linear systems.

<u>Obs</u>: Let S be a subspace. Any two basis of S have the same number of elements. This number is

Called the dimension of S.