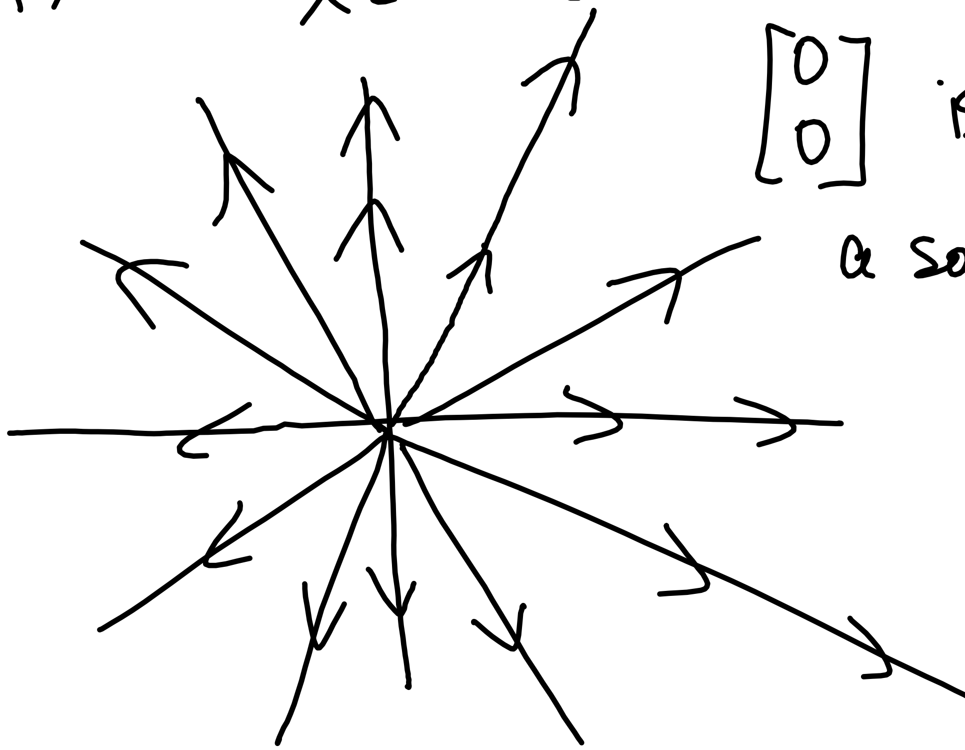
Phase plane for linear systems

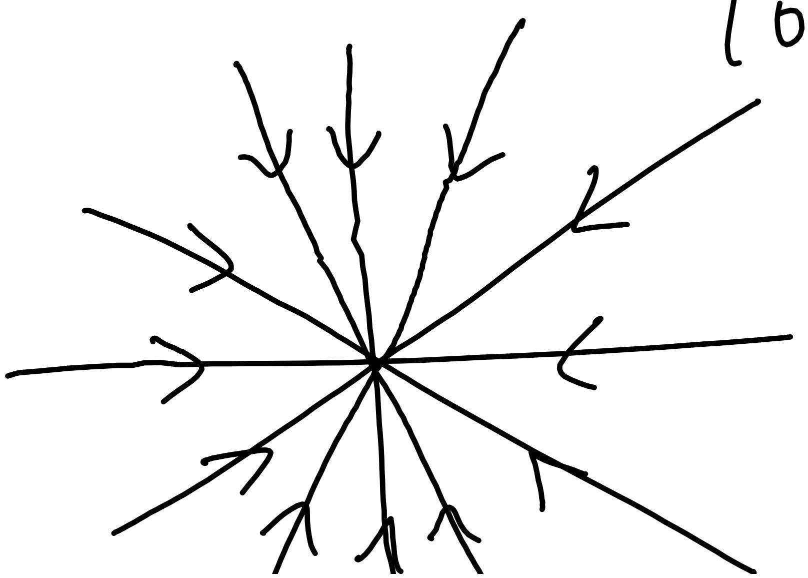
Case 1 $A = \lambda I \in \mathbb{R}^{2\times 2}$ $\lambda > 0$

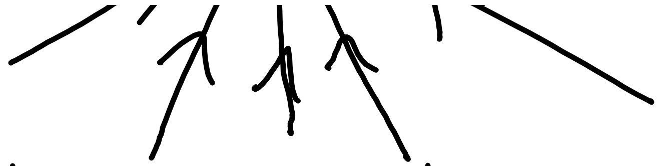
$x' = Ax$ $x = u\, e^{\lambda t}$

$\begin{bmatrix}0\\0\end{bmatrix}$ is a source



Case 2 $A = \lambda I$ $\lambda < 0$ $\begin{bmatrix}0\\0\end{bmatrix}$ is a sink

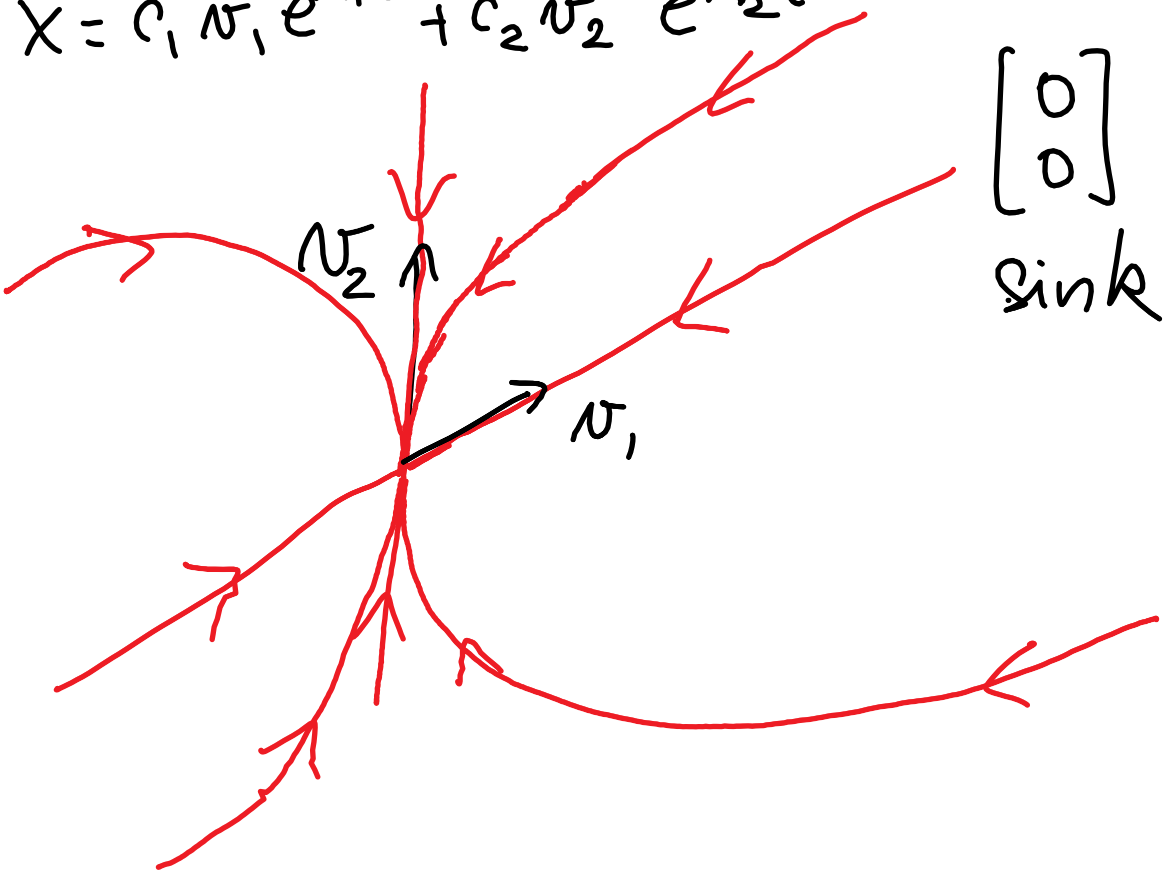




Case 3 A has two real eigenvalues

$\lambda_1 < \lambda_2 < 0$. $\quad A v_i = \lambda_i v_i \quad v_i \neq 0$

$x = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$



$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a sink

$c_1 = c_2 = 1$

$$x = e^{\lambda_2 t}\Big(v_1 \underbrace{e^{(\lambda_1 - \lambda_2)t}}_{\to 0} + v_2\Big)$$

as $t \to +\infty$

$$x = e^{\lambda_1 t}\Big(v_1 + \underbrace{v_2 e^{(\lambda_2 - \lambda_1)t}}_{\to 0}\Big)$$

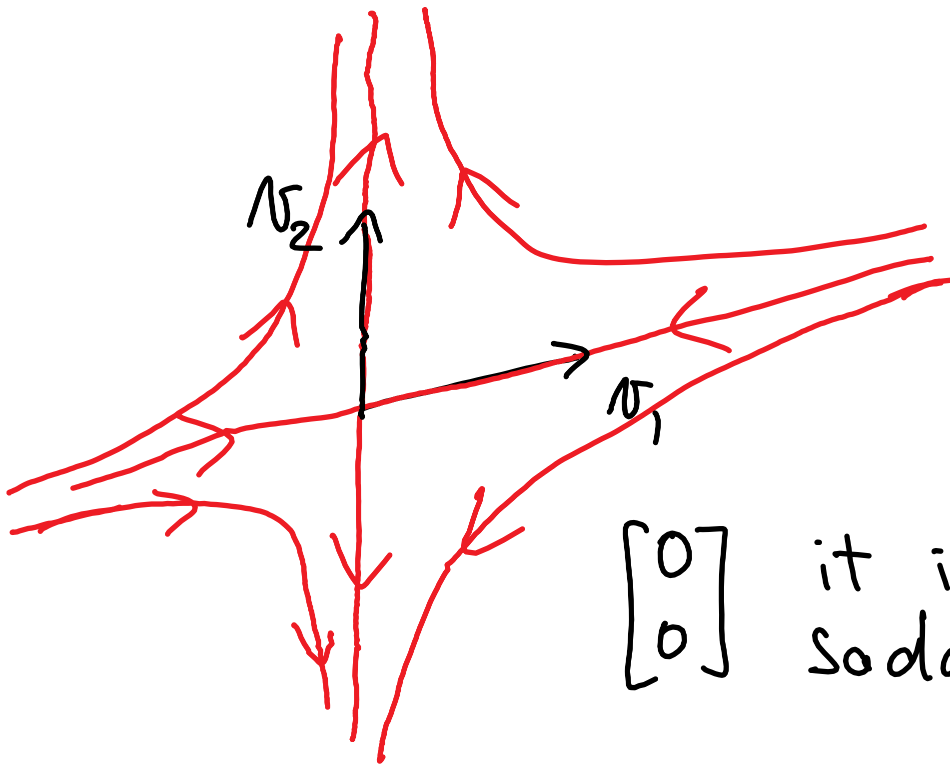
as $t \to -\infty$

$\lambda_1 < \lambda_2 \Rightarrow \lambda_2 - \lambda_1 > 0$

$\underbrace{\qquad\qquad}_{\to 0}$ as $t \to -\infty$

$\lambda_1 < \lambda_2 \Rightarrow \lambda_2 - \lambda_1 > 0$

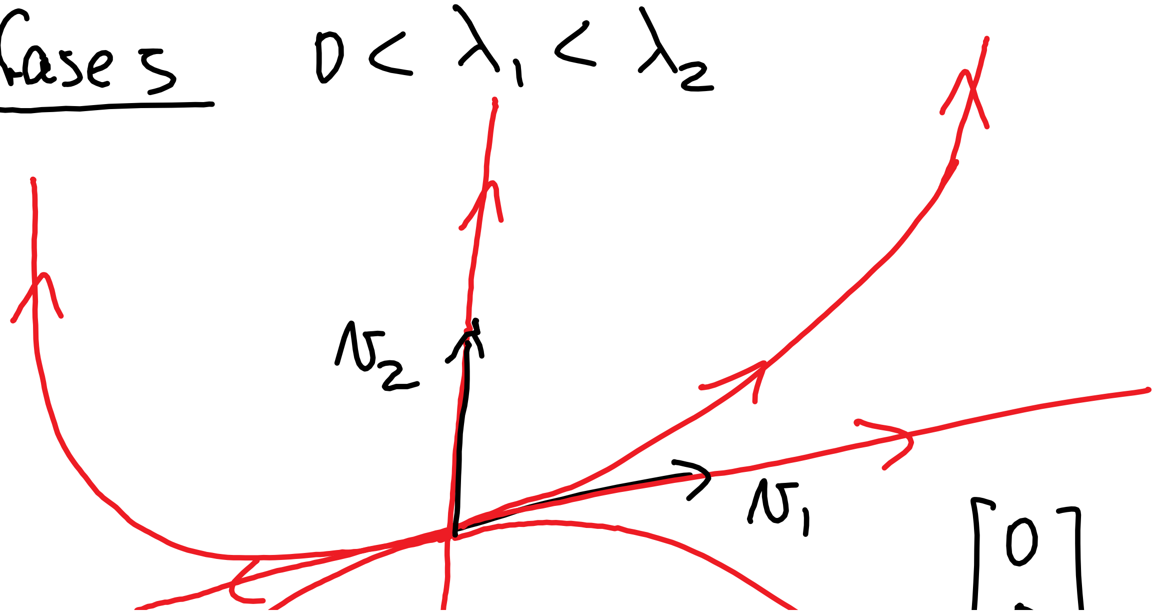
Case 4 $\quad \lambda_1 < 0 < \lambda_2 \qquad A v_i = \lambda_i v_i$



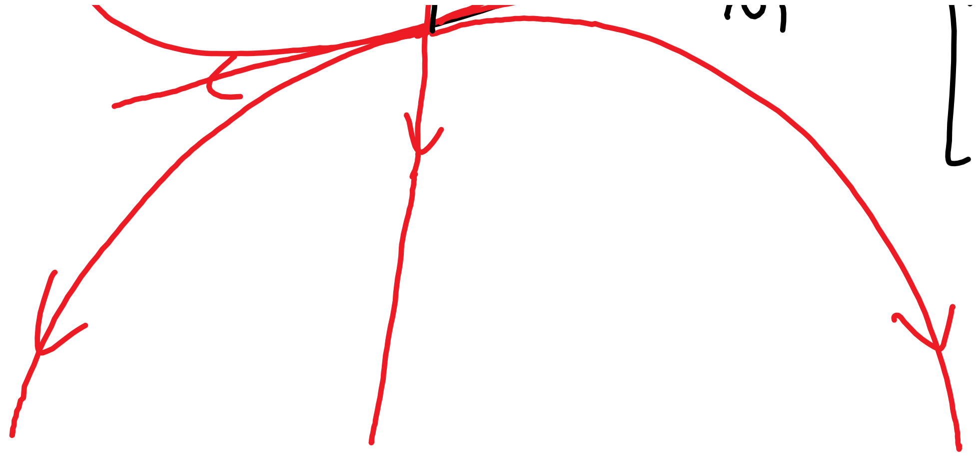
$\begin{bmatrix}0\\0\end{bmatrix}$ it is a saddle

$$x = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

Case 5 $\quad 0 < \lambda_1 < \lambda_2$



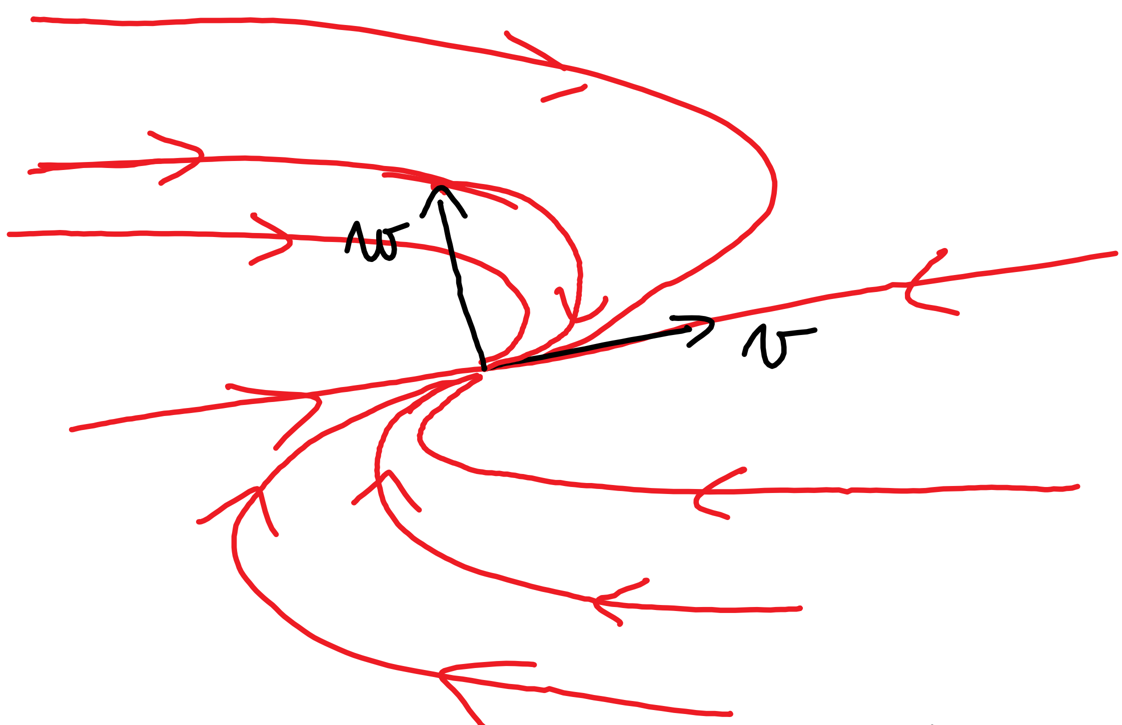
$\begin{bmatrix}0\\ \cdot\end{bmatrix}$ is a



$\begin{bmatrix}0\\0\end{bmatrix}$ is a source

<u>Case 6</u>: λ is the only eigenvalue of A, $A \neq \lambda I$ & $\lambda < 0$

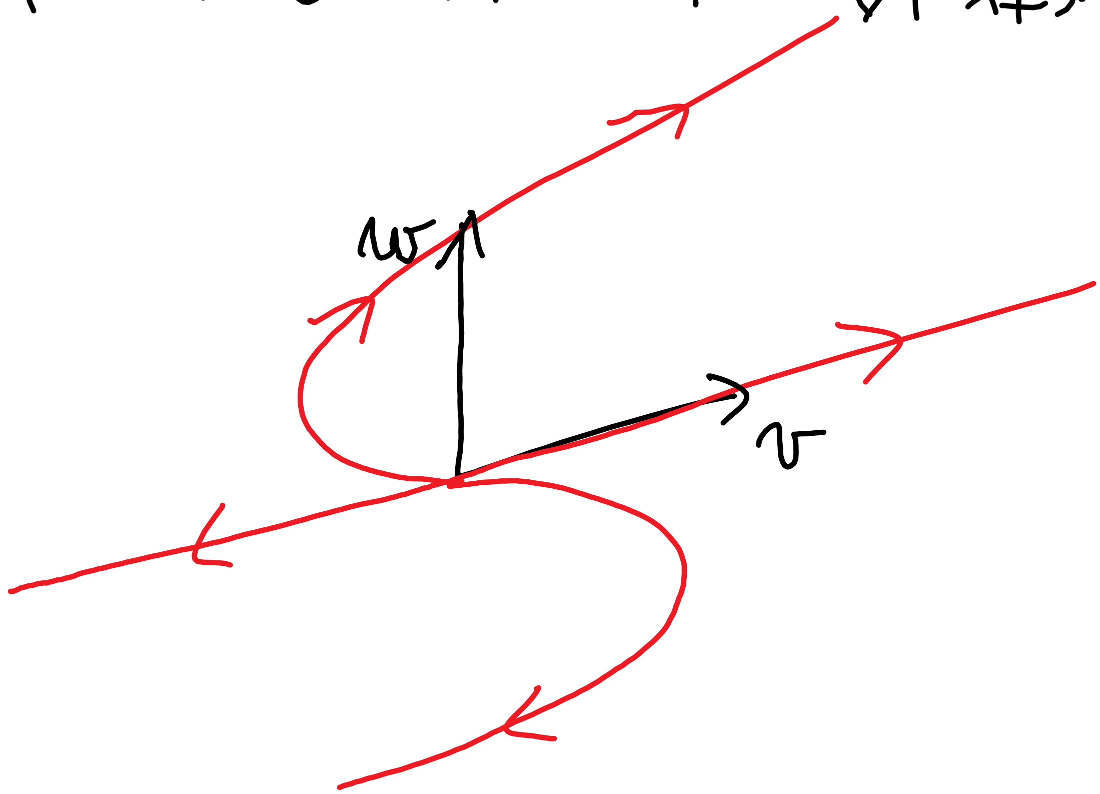
$Av = \lambda v \quad v \neq 0 \quad (A-\lambda I)w = v$



$c_1 e^{\lambda t} v + c_2 (tv + w) e^{\lambda t}$

$c_1 = 0 \quad c_2 = 1 \qquad \underbrace{t e^{\lambda t}}_{\substack{\to 0 \\ t \to \infty}} \left(v + \frac{w}{t} \right)$

<u>Case 7</u> $Av = \lambda v$ λ only eigenvalue of A $\quad \lambda > 0 \quad A \neq \lambda I \quad (A - \lambda I) w = v$



<u>Case 8</u> $Av = \lambda v \quad \lambda \notin \mathbb{R} \quad v \neq 0.$

$v = v_1 + i v_2 \quad v_1, v_2 \in \mathbb{R}^2 \quad \lambda = \alpha + \beta i$

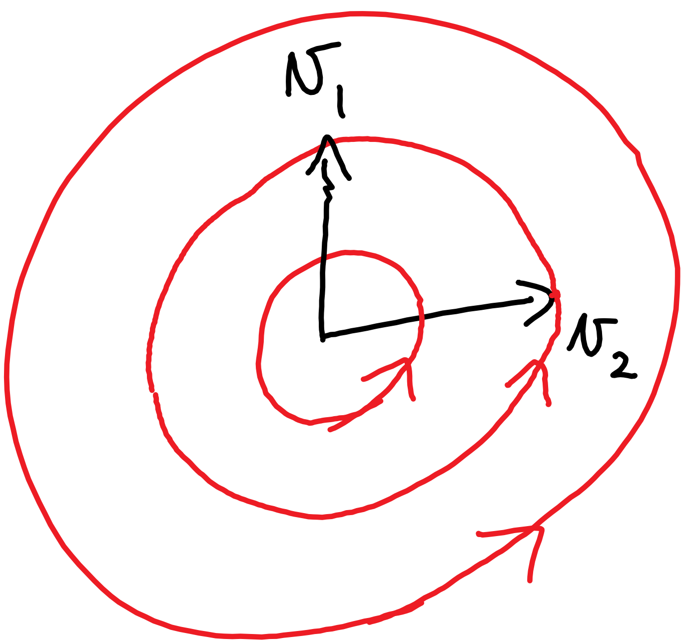
$\alpha, \beta \in \mathbb{R}.$

$e^{(\alpha + i\beta)t} (v_1 + i v_2) = e^{\alpha t} \left(\cos(\beta t) v_1 - \sin(\beta t) v_2 \right)$

$$+ i e^{\alpha t} (\cos(\beta t) v_2 + \sin(\beta t) v_1)$$

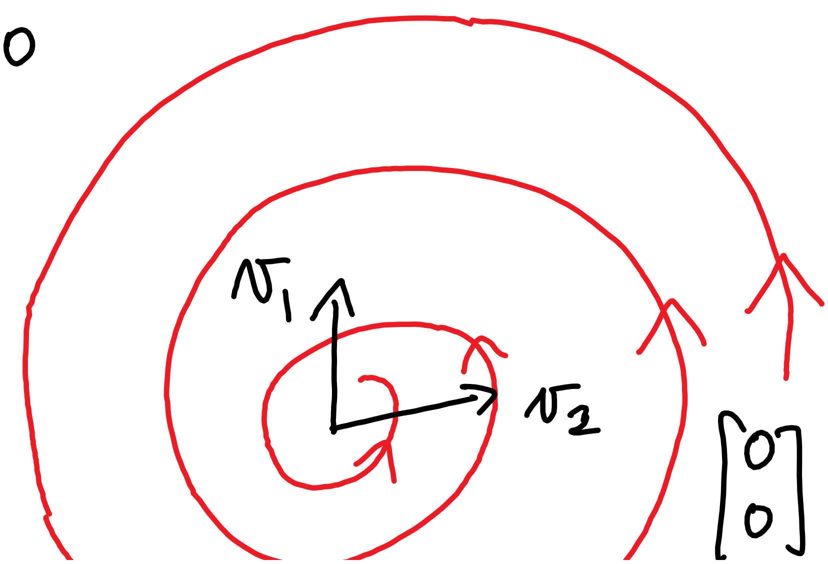
$$e^{\alpha t} (\cos(\beta t) v_2 + \sin(\beta t) v_1)$$

Case $\alpha = 0$



$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a center

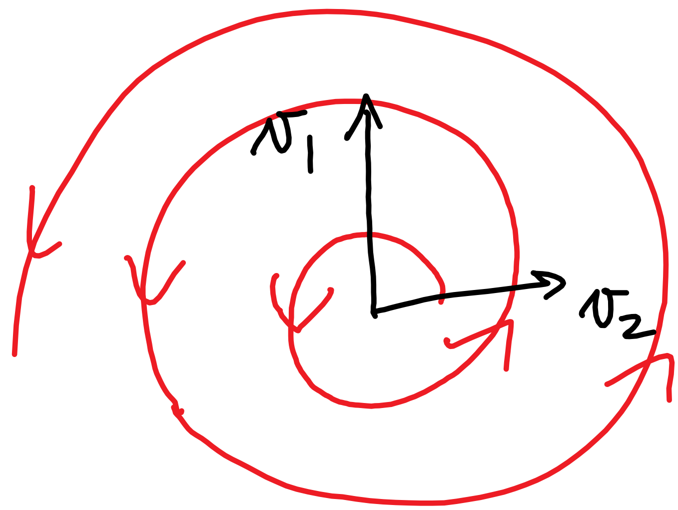
Case $\alpha < 0$



$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a stable spiral

Case $\alpha > 0$



$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ unstable spiral

Def: $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{R}^2$ is an equilibrium solution or a fixed point of $x' = Ax$.

This is because $x(t) = 0 \Rightarrow \underset{0}{\underset{\parallel}{x'}} = \underset{0}{\underset{\parallel}{Ax}}$

Def: 0 is a stable fixed point of

of $x' = Ax$ if $x(t)$ remains close to 0 for all $t > 0$ whenever $x(0)$ was already close to 0.

0 is unstable if it is not stable

Obs: Stable fixed points: sinks, centers, stable spirals

Unstable fixed points: sources, saddles, unstable spirals.

Th: Unstable if $Re(\lambda) > 0$ for one eigenvalue

stable if $Re(\lambda) \le 0$ for both eigen-values.

Obs: We looked at the cases where 0 was the only fixed point. This is the case when A has an inverse =

$\det A \neq 0$