4541_7

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$x' = Ax \qquad A \in \mathbb{R}^{2\times 2}$

We need to find two solutions, x_1 and x_2, such that $x_1(0)$ and $x_2(0)$ are linearly independent. Any other solution is a linear combination of $x_1(0)$ and $x_2(0)$

<u>Case 1</u>: $A = \lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$x_1' = \lambda x_1 \longrightarrow x_1 = c_1 e^{\lambda t}$

$x_2' = \lambda x_2 \longrightarrow x_2 = c_2 e^{\lambda t}$

$x = x(0) e^{\lambda t}$

<u>Obs</u>: Let $A \in \mathbb{R}^{2\times 2}$, $v \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$.

If $Av = \lambda v$ then $x = v e^{\lambda t}$ is a solution of $x' = Ax$

<u>proof</u>: $\boxed{x'} = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \frac{d}{dt} v_1 e^{\lambda t} \\ \frac{d}{dt} v_2 e^{\lambda t} \end{bmatrix} = \begin{bmatrix} \lambda v_1 e^{\lambda t} \\ \lambda v_2 e^{\lambda t} \end{bmatrix}$

$$\ldots \begin{bmatrix} \lambda_2 \end{bmatrix} \quad \begin{bmatrix} \frac{d}{dt} v_2 e^{\lambda t} \end{bmatrix} \quad \begin{bmatrix} \lambda v_2 e^{\lambda t} \end{bmatrix}$$

$$= \lambda v e^{\lambda t} = \lambda x$$

Example: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad \lambda = 3 \quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$x = v e^{\lambda t} = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

$$x' = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} = Ax \checkmark$$

Back on solving $x' = Ax$

Case 2: A has 2 different real eigenvalues α and β with eigenvectors v and w. $Av = \alpha v$ and $Aw = \beta w$

From the last observation, $x_1 = v e^{\alpha t}$ and $x_2 = w e^{\beta t}$ are solutions of $x' = Ax$.

Since $x_1(0) = v$ and $x_2(0) = w$ are

eigenvectors of different eigenvalues, then, they are linearly independent.

Thus, x is a solution of $x' = Ax$ if and only if

$$x = c_1 v e^{\alpha t} + c_2 w e^{\beta t}$$

Example: $x_1' = 4x_1 - 2x_2 \qquad x_1(0) = 1$

$x_2' = 3x_1 - x_2 \qquad x_2(0) = 2$

$$A = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$$

$$P(\lambda) = \det \begin{bmatrix} 4-\lambda & -2 \\ 3 & -1-\lambda \end{bmatrix} = (4-\lambda)(-1-\lambda) + 6 =$$

$$= \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$$

Eigenvectors $\boxed{\lambda_1 = 1}$ $\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}$ $v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\boxed{\lambda_2 = 2}$ $\begin{bmatrix} 2 & -2 \\ \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ \end{bmatrix}$

$\boxed{\lambda_2 = 2}$ $\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$x = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

Impose the initial conditions. Set $t=0$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$1 = 2c_1 + c_2$
$2 = 3c_1 + c_2$

$c_1 = 1 \quad c_2 = -1$

$$x = \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{t} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

$x_1 = 2e^{t} - e^{2t}$
$x_2 = 3e^{t} - e^{2t}$

<u>Obs</u>: Assume A has only one eigenvalue λ and $A \neq \lambda I$. Let v be an eigenvector. Let w be a solution of $(A - \lambda I) w = v$. Then $x = (w + tv) e^{\lambda t}$ is a solution of $x' = Ax$

<u>proof</u>: $x' = v e^{\lambda t} + \lambda (w + tv) e^{\lambda t}$

$Ax = A(w + tv)e^{\lambda t} = (\lambda w + v + t\lambda v)e^{\lambda t}$

then $x' = Ax$ ✓ [auxiliary notes:

$(A - \lambda I)w = v$ then $Aw = \lambda w + v$)

<u>Case 3</u>: A has only 1 eigenvalue λ.

v is an eigenvector. w a solution of

$(A - \lambda I)w = v$. then $x' = Ax \iff$

$$x = c_1 v e^{\lambda t} + c_2 (w + tv) e^{\lambda t}$$

for some c_1 and $c_2 \in \mathbb{R}$.

<u>Example</u>: $x_1' = 6x_1 - x_2$

$x_2' = x_1 + 4x_2$

$A = \begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix}$ $\quad P(\lambda) = \det \begin{bmatrix} 6-\lambda & -1 \\ 1 & 4-\lambda \end{bmatrix} =$

$= \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2 \quad \lambda = 5$

Eigenvector $\boxed{\lambda = 5}$ $\quad \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = A - 5I \quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$(A-5I)w = v \quad \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)e^{5t}$$

$x_1 = c_1 e^{5t} + c_2(1+t)e^{5t}$

$x_2 = c_1 e^{5t} + c_2 t e^{5t}$

Complex valued functions

$f(t) = f_1(t) + i f_2(t) \qquad t, f_1(t) \,\&\, f_2(t)$ real

Ex: $f(t) = t + ie^t$

Def: $a, b \in \mathbb{R} \quad z = a+bi$

$$e^z = e^{a+bi} = e^a(\cos b + i\sin b)$$

Ex: $e^{i\pi} = e^0(\cos\pi + i\sin\pi) = -1$

$$e^{1+i\frac{\pi}{2}} = e\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = ie$$

Properties: 0) $a \in \mathbb{R}$

$$e^{a+i0} = e^a(\cos 0 + i\sin 0)\ e^a \quad \checkmark$$

1) $$e^{z_1+z_2} = e^{z_1}\, e^{z_2}$$

Ex: $\lambda = a+bi$, $a, b \in \mathbb{R}$, $t \in \mathbb{R}$

$$f(t) = e^{\lambda t} = e^{at+bti} = e^{at}(\cos(bt) + i\sin(bt))$$

$$= f_1(t) + i f_2(t)$$

$$\mathrm{Re}(f(t)) = f_1(t) = e^{at}\cos(bt)$$

$$\mathrm{Im}(f(t)) = f_2(t) = e^{at}\sin(bt)$$

Derivatives of complex valued functions

$$f(t) = f_1(t) + i f_2(t)$$

$$f'(t) = f_1'(t) + i f_2'(t)$$

Ex: $$\frac{d}{dt}(t + it^2) = 1 + i2t$$

Ex: $\lambda = a+bi$

$$\frac{d}{dt}(e^{\lambda t}) = \frac{d}{dt}(e^{at}\cos bt) + i\frac{d}{dt}(e^{at}\sin bt) =$$

$$\boxed{\frac{d}{dt}(e^{\lambda t})} = \frac{d}{dt}(e^{at}\cos bt) + i\,\frac{d}{dt}(e^{at}\sin bt) =$$

$$= \boxed{\lambda e^{\lambda t}} \quad \checkmark$$

<u>Obs</u>: $Av = \lambda v$. $A \in \mathbb{R}^{2\times 2}$, $\lambda \in \mathbb{C}$, $v \in \mathbb{C}^2$. then $x = v e^{\lambda t}$ is a solution of $\quad x' = Ax$

<u>Example</u>: $\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 \end{aligned} \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$P(\lambda) = \det\begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 + 1 \qquad \lambda = \pm i$$

$$(A - iI)v = 0 \qquad \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$x = v e^{\lambda t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{it} \qquad \begin{aligned} x_1 &= e^{it} \\ x_2 &= i e^{it} \end{aligned}$$

Check. $\quad x_1' = i e^{it} = x_2$

Check: $x_1' = ie^{it} = x_2$

$x_2' = -e^{it} = -x_1$ ✓

Obs: If x is a complex solution of $x' = Ax$, then $\text{Re}(x)$ and $\text{Im}(x)$ are real solutions of $x' = Ax$

proof: $x = \text{Re}(x) + i\,\text{Im}(x)$

$$\frac{dx}{dt} = \frac{d}{dt}\text{Re}(x) + i\frac{d}{dt}\text{Im}(x)$$

$$\begin{array}{ccccc} \frac{dx}{dt} & = & \frac{d}{dt}\text{Re}(x) & + i & \frac{d}{dt}\text{Im}(x) \\ \| & & \| & \| & \| \\ Ax & = & A\,\text{Re}(x) & + i & A\,\text{Im}(x) \end{array}$$

Example: $x_1' = x_2$ $\quad x = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{it}$

$x_2' = -x_1$

$$x = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{it} = \begin{bmatrix} e^{it} \\ ie^{it} \end{bmatrix} = \begin{bmatrix} \cos t + i\sin t \\ i(\cos t + i\sin t) \end{bmatrix} =$$

$$= \begin{bmatrix} \cos t + i\sin t \\ \end{bmatrix}$$

$$= \begin{bmatrix} \cos t + i\sin t \\ -\sin t + i\cos t \end{bmatrix}$$

$$Re(x) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} = y \qquad y_1' = -\sin t = y_2, \quad y_2' = -\cos t = -y_1 \ \checkmark$$

$$Im(x) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} = z \qquad z_1' = \cos t = z_2, \quad z_2' = -\sin t = -z_1$$

<u>Obs</u>: Let $A \in \mathbb{R}^{2\times 2}$. If v is an eigenvector of A with eigenvalue λ, and $\lambda \notin \mathbb{R}$, then $Re(v)$ and $Im(v)$ are linearly independent

<u>Example</u> $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$P(\lambda) = \det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 + 1 = 0 \Rightarrow \lambda = 1 \pm i$$

Eigenvector of $\lambda = 1+i$ $\quad A - (1+i)I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$

$v = \begin{bmatrix} i \\ 1 \end{bmatrix}$ $\quad Re(v) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $Im(v) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are linearly independent.

<u>Last case</u>: $x' = Ax$. λ is an eigenvalue of A and $\lambda \notin \mathbb{R}$. Let v be an eigenvector of A with eigenvalue λ. Then x is a solution of $x' = Ax$ if and only if, x is of the form

$$x = c_1 \operatorname{Re}(v e^{\lambda t}) + c_2 \operatorname{Im}(v e^{\lambda t})$$

for some c_1 and $c_2 \in \mathbb{R}$

<u>Example</u> Solve the IVP

$$x_1' = x_1 - x_2 \qquad x_1(0) = -1$$

$$x_2' = x_1 + x_2 \qquad x_2(0) = 2$$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad P(\lambda) = \det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 + 1$$

$$(\lambda - 1)^2 + 1 = 0 \qquad \lambda = 1 \pm i$$

$$\lambda = 1+i \qquad A - (1+i)I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \qquad v = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$v e^{\lambda t} = \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(1+i)t} = \begin{bmatrix} i \\ 1 \end{bmatrix} e^{t}(\cos t + i \sin t)$$

$$= \begin{bmatrix} -e^t \sin t + i e^t \cos t \\ e^t \cos t + i e^t \sin t \end{bmatrix} = \begin{bmatrix} -e^t \sin t \\ e^t \cos t \end{bmatrix} + i \begin{bmatrix} e^t \cos t \\ e^t \sin t \end{bmatrix}$$

$$x = c_1 e^t \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

Replace t by 0 to get

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{aligned} c_1 &= 2 \\ c_2 &= -1 \end{aligned}$$

$$x = 2e^t \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} - e^t \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$x_1 = -2e^t \sin t - e^t \cos t$$

$$x_2 = 2e^t \cos t - e^t \sin t$$