Claim: $A \in \mathbb{R}^{2\times 2}$. A has only one eigenvalue λ. $A \neq \lambda I$. Let v be an eigenvector of A. Then, there exists w such that

$$(A-\lambda I)w = v$$

Example: $A = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix}$

$$P(\lambda) = \det \begin{bmatrix} 3-\lambda & 4 \\ -1 & 7-\lambda \end{bmatrix} = \lambda^2 - 10\lambda + 25$$

$\lambda = 5$ is the only eigenvalue and

$A \neq 5I$

Eigenvector $A - 5I = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$ $\quad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Find w $\quad (A-\lambda I)\, w = v$

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad w = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Linear independence

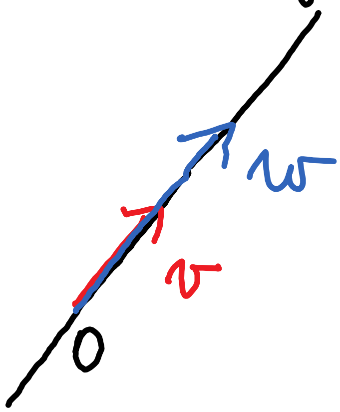
Def: Two vectors v & $w \in \mathbb{R}^2$ are linearly dependent if one is a multiple of the other. they are linearly independent, if they are not linearly dependent.

Ex: 1) $\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ are linearly dependent

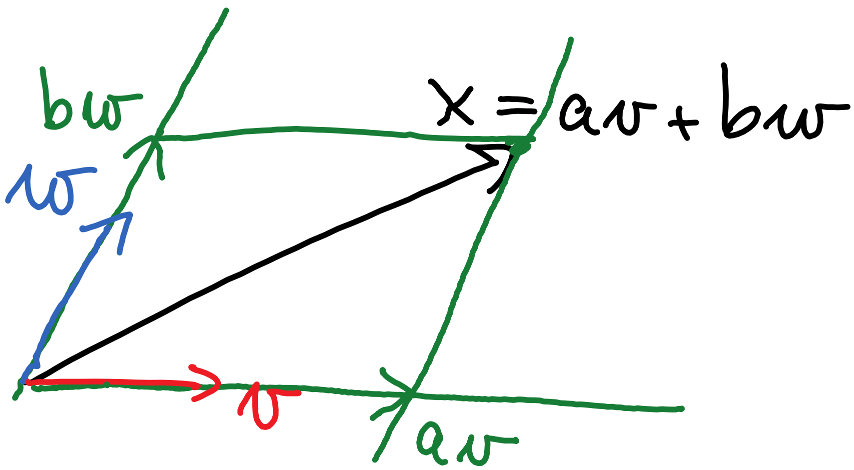
2) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are linearly independent

3) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ are linearly dependent.

Geometry: Two vectors are linearly dependent if they both belong to a dependent if they both belong to a straight line that contains the origin.



<u>Obs</u>:



Let $v, w \in \mathbb{R}^2$ linearly independent.
Let $x \in \mathbb{R}^2$. There exists unique numbers a and b such that $x = av + bw$

<u>Def</u>: We say that x is a linear combination of v & w if $x = av + bw$ for some numbers a & b.

<u>Obs</u>: Let v and $w \in \mathbb{R}^2$. $x \in \mathbb{R}^2$

Finding a and b such that $av + bw = x$.

$$av + bw = \begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We need to solve the system of equations

$$\begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The unknowns are a & b.

<u>Example</u>: $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad w = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad x = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$\begin{array}{cc} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \\ v \quad w \end{array} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{array}{c} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\ x \end{array}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{(-1)} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$a = 1$ $\quad b = -1$ $\quad$ check $\quad 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

a v b w x

<u>Inverse</u>: $A \in \mathbb{R}^{2\times 2}$. the inverse of A is the matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$

note: not every matrix has an inverse

note: A has an inverse $\Leftrightarrow$ $\det A \neq 0$.

In this case, $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

<u>Example</u> $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{(-1)} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

<u>Check</u>: $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

<u>Obs</u>: $A \in \mathbb{R}^{2\times 2}$. then $\det(A) \neq 0 \Leftrightarrow$ its columns are linearly independent

Example

1) $A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$ $\det(A) = 2 - 2 = 0$

columns linearly dependent, determinant equal to zero

2) $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $\det(A) = 1$

columns linearly independent, determinant $= 0$

Obs: $A = [v \ w] = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$

$$A \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a v_1 + b w_1 \\ a v_2 + b w_2 \end{bmatrix} =$$

$$a \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + b \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = a v + b w$$

Claims: 1) Eigenvectors of different eigenvalues are linearly independent.

<u>Example</u> $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

	Eigenvectors
	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2) $A \in \mathbb{R}^{2\times 2}$. A has only one eigenvalue λ. $A \neq \lambda I$. v an eigenvector. w a solution of $(A-\lambda I)w = v$. Then v and w are linearly independent

<u>proof</u>: If $w = \alpha v$, then $v = (A-\lambda I)w = (A-\lambda I)\alpha v = \alpha(A-\lambda I)v = \alpha 0 = 0$. Contradiction

<u>Example</u>: $A = \begin{bmatrix} 3 & 4 \end{bmatrix}$ $\quad \lambda = 5$

<u>Example</u>: $A = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix}$ $\lambda = 5$

$v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $w = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $(A-5I)v = 0$

$(A-5I)w = v$

$\mathrm{Re}(3+2i) = 3$

$\mathrm{Re}\left(\begin{bmatrix} 3+2i \\ -1-5i \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

<u>Def</u>: $v \in \mathbb{C}^2$ $\mathrm{Re}(v) = \begin{bmatrix} \mathrm{Re}(v_1) \\ \mathrm{Re}(v_2) \end{bmatrix}$

and $\mathrm{Im}(v) = \begin{bmatrix} \mathrm{Im}(v_1) \\ \mathrm{Im}(v_2) \end{bmatrix}$

<u>Obs</u>: $A \in \mathbb{R}^{2\times 2}$. $\lambda \in \mathbb{C}$. $v \in \mathbb{C}^2$

$v \neq 0$. $Av = \lambda v$. $\mathrm{Im}(\lambda) \neq 0$.

then $\mathrm{Re}(v)$ and $\mathrm{Im}(v)$ are linearly independent.

Example $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Find the eigen-values & eigenvectors

$$P(\lambda) = \det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = (\lambda-1)^2 + 1 = 0$$

$$\lambda = 1 \pm i$$

Eigenvector of $\lambda = 1+i$

$$A - (1+i) I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \qquad v = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\mathrm{Re}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathrm{Im}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Linear systems of 2 differential equations

these are systems of the form

$$x_1' = a_{11} x_1 + a_{12} x_2$$

$$x_2' = a_{21} x_1 + a_{22} x_2$$

$a_{ij} \in \mathbb{R}$ are given. Goal: Find $x_1(t)$ & $x_2(t)$ that satisfy the equations.

Example
$$x_1' = 2x_1 + x_2$$
$$x_2' = -x_1$$

In matrix form: $x' = Ax$

In the abov example, $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

Fact: If $x' = Ax$ and $y' = Ay$, a and $b \in \mathbb{R}$. then $z = ax + by$ is also a solution

proof: $\boxed{z'} = ax' + by' = aAx + bAy =$
$= A(ax) + A(by) = A(ax+by) = \boxed{Az}$

Theorem: $A \in \mathbb{R}^{2\times 2}$. If x_1 & x_2 are solutions of $x' = Ax$ and

$x_1(0)$, $x_2(0)$ are linearly independent, then x is a solution $\Longleftrightarrow$ x is a linear combination of x_1 and x_2

<u>proof</u>: Recall: For every $u \in \mathbb{R}^2$ there exist a unique solution of

$$(*) \begin{cases} x' = Ax \\ x(0) = u \end{cases}$$

I want to show that $\exists$ c_1 and c_2 such that $x = c_1 x_1 + c_2 x_2$, where x is the unique solution of $(*)$.

Since $x_1(0)$ and $x_2(0)$ are linearly independent, $\exists$ c_1 and c_2 such that

$$u = c_1 x_1(0) + c_2 x_2(0)$$

then, $x = c_1 x_1 + c_2 x_2$ satisfies $x' = Ax$ because x_1 & x_2 do, and also

$x(0) = c_1 \, x_1(0) + c_2 \, x_2(0) = u$. Then

$c_1 x_1 + c_2 x_2$ is the unique solution of $\circledast$