Example: Find the eigenvalues and eigenvectors of $\begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix}$

$$\det \begin{bmatrix} 3-\lambda & 4 \\ -1 & 7-\lambda \end{bmatrix} = (3-\lambda)(7-\lambda)+4 = \lambda^2-10\lambda+25=0$$

$\lambda=5$

$A-5I = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$ $\quad v=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Eigenvalues	Eigenvectors
5	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Complex numbers

i is a ~~new~~ number $\quad i^2=-1$

Def: A complex number is a number of the form $a+bi$ where a and b are real

Example: $2+3i$

Operations with complex numbers

Addition: $(a+bi)+(c+di)=(a+c)+(b+d)i$

Ex: $(-2+3i)+(1-2i)=-1+i$

Multiplication $(a+bi)(c+di)=(ac-bc)+i(ad+bc)$

Ex: $(-2+3i)(1-2i)=-2+4i+3i+6=4+7i$

Def: 1) $\mathbb{C}$ denotes the set of complex numbers

2) $z\in\mathbb{C}$, $z=a+bi$ with $a,b\in\mathbb{R}$

$a=\mathrm{Re}(z)$ real part of z

$b=\mathrm{Im}(z)$ imaginary part of z

$z=\mathrm{Re}(z)+i\,\mathrm{Im}(z)$

Ex: $\mathrm{Re}(-2+3i)=-2$ & $\mathrm{Im}(-2+3i)=3$

Obs: $x^2+1=0$ does not have any

real roots.

$x^2 = -1$ then $x = i$ or $x = -i$

Every polynomial (with real coefficients) has at least one root if we allow the roots to be complex numbers

<u>Roots of polynomials of degree 2</u>

$ax^2 + bx + c = 0 \qquad a, b, c \in \mathbb{R} \quad a \neq 0$

$\Delta = b^2 - 4ac$

<u>Case 1</u>: $\Delta > 0$ the roots are

$$x = \frac{-b + \sqrt{\Delta}}{2a} \text{ and } x = \frac{-b - \sqrt{\Delta}}{2a}$$

(two different roots)

<u>Case 2</u>: $\Delta = 0$ there is only one root $x = -\frac{b}{2a}$

<u>Case 3</u> $\Delta < 0$ there are two roots that are complex numbers

that are complex numbers

$$x = \frac{-b + i\sqrt{-\Delta}}{2a} \quad \text{and} \quad x = \frac{-b - i\sqrt{-\Delta}}{2a}$$

<u>Example</u>: $x^2 + 2x + 2 = 0$

$$x = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm i\sqrt{4}}{2} = -1 \pm i$$

<u>Def</u>: $z = a + bi \quad a, b \in \mathbb{R}$. the complex conjugate of z is

$$\bar{z} = a - bi$$

<u>Example</u> $\overline{3+2i} = 3-2i$

<u>Obs</u>: 1) $(3+2i) + (5-i) = 8+i$

$$\overline{(3+2i)+(5-i)} = 8-i$$

$$\overline{(3+2i)} + \overline{(5-i)} = 3-2i + 5+i = 8-i$$

1) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

2) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

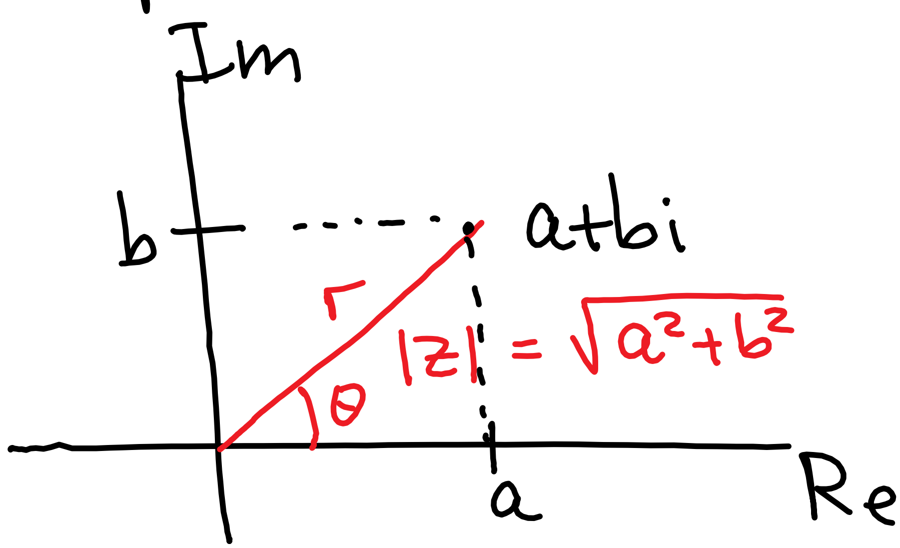
3) $\overline{z} = z$ if and only if $z \in \mathbb{R}$

<u>Obs</u>: $z = a+bi$, $a, b \in \mathbb{R}$

$z\overline{z} = (a+bi)(a-bi) = a^2-(bi)^2 = a^2+b^2$

<u>Def</u>: The modulus or absolute value of a complex number $z = a+bi$ $(a,b\in\mathbb{R})$ is $|z| = \sqrt{a^2+b^2} = \sqrt{z\overline{z}}$

<u>Complex plane</u>



$\theta = \arg(z)$ $\quad a = r\cos\theta$ $\quad r = \sqrt{a^2+b^2}$

$r = |z|$ $\quad b = r\sin\theta \quad \tan\theta = \frac{b}{a}$

$Re(z) = |z| \cos(\arg(z)) \quad |z| = \sqrt{Re(z)^2 + Im(z)^2}$

$Im(z) = |z| \sin(\arg(z)) \quad \tan(\arg(z)) = \frac{Im(z)}{Re(z)}$

Obs: Let z & $w \in \mathbb{C}$ $\quad w \neq 0$

$$\frac{z}{w} = \frac{z\,\overline{w}}{w\,\overline{w}} = \frac{z\,\overline{w}}{|w|^2}$$

Ex: $\frac{3+2i}{2-i} = \frac{(3+2i)(2+i)}{(2-i)(2+i)} = \frac{(6-2)+7i}{5} = \frac{4}{5} + \frac{7}{5}i$

Obs: Let $P(x)$ be a real polynomial.

If $P(z)=0$, then $P(\overline{z}) = 0$

proof: $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.

$a_0, a_1, \ldots, a_n \in \mathbb{R}$

$$\overline{P(z)} = \overline{a_n z^n + \cdots + a_1 z + a_0} =$$

$$= \overline{a}_n \overline{z}^n + \cdots + \overline{a}_1 \overline{z} + \overline{a}_0 =$$

$= a_n \bar{z}^n + \cdots + a_1 \bar{z} + a_0 = P(\bar{z})$

Thus, if $P(z) = 0$, then $P(\bar{z}) = \overline{P(z)} = \bar{0} = 0$

also.

<u>Example</u> $x^2 + 2x + 2 = 0$ the two solutions

are $-1+i$ & $-1-i$. Note that $-1-i = \overline{-1+i}$

<u>Def</u>: $\mathbb{C}^2$ is the set of 2-vectors whose components are complex numbers

$z \in \mathbb{C}^2$ means $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ with

z_1 and $z_2 \in \mathbb{C}$.

<u>Example</u> $\begin{bmatrix} 1-i \\ 2+3i \end{bmatrix} \in \mathbb{C}^2$ (complex vector)

<u>Def</u>: $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{C}^2$ then $\bar{z} = \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix}$

<u>Obs</u>: $A \in \mathbb{R}^{2\times 2}$. $z \in \mathbb{C}^2$. $\lambda \in \mathbb{C}$

$\overline{Az} = \bar{A}\,\bar{z} = A\,\bar{z}$ $\qquad$ $\overline{\lambda z} = \bar{\lambda}\,\bar{z}$

<u>Obs</u>: $A \in \mathbb{R}^{2\times 2}$. $\lambda \in \mathbb{C}$ & $v \in \mathbb{C}^2$ such that $Av = \lambda v$. then $A\bar{v} = \bar{\lambda}\bar{v}$

If v is an eigenvector of A with eigenvalue λ, then $\bar{v}$ is also an eigenvector of A with eigenvalue $\bar{\lambda}$

We allow eigenvalues and eigenvectors to be complex

<u>Example</u>: Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$

$$P(\lambda) = \det \begin{bmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{bmatrix} = \lambda^2 - 10\lambda + 29$$

$P(\lambda) = 0$ $\quad$ $\lambda = \dfrac{10 \pm \sqrt{100 - 4(29)}}{2} = \dfrac{10 \pm i\sqrt{16}}{2}$

$\lambda = 5 \pm 2i$

Eigenvector of $\lambda = 5+2i$

$$A-(5+2i)I = \begin{bmatrix} 1-2i & -1 \\ 5 & -1-2i \end{bmatrix}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \begin{bmatrix} 1-2i & -1 \\ 5 & -1-2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v = \begin{bmatrix} 1 \\ 1-2i \end{bmatrix}$$

Eigenvector of $5-2i$

$$A-(5-2i)I = \begin{bmatrix} 1+2i & -1 \\ 5 & -1-2i \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 1+2i \end{bmatrix}$$

Eigenvalues	Eigenvectors
$5+2i$	$\begin{bmatrix} 1 \\ 1-2i \end{bmatrix}$
$5-2i$	$\begin{bmatrix} 1 \\ 1+2i \end{bmatrix}$

<u>Obs</u>: $A \in \mathbb{R}^{2\times 2}$. Then one of the following happens

1) A has 2 distinct real eigenvalues
2) A has 1 real eigenvalue.
3) A has 2 complex (non-real) eigenvalue and they are complex conjugate of each other.

<u>Claim</u>: $A \in \mathbb{R}^{2\times 2}$. Assume A has only one eigenvalue λ. Assume $A \neq \lambda I$

Let v be an eigenvector of A. $Av = \lambda v$

then

$$(A - \lambda I) w = v$$

has a solution for w.