$\boxed{1.5}$ Replace t by s to get

$$\frac{\partial}{\partial s}\phi(s,u) = f(s,\phi(s,u))$$

integrate over s from 0 to t

$$(1)\quad \phi(t,u) = u + \int_0^t f(s,\phi(s,u))\,ds$$

Take $\frac{\partial}{\partial u}$ of eq(1)

$$(2)\quad \frac{\partial\phi}{\partial u}(t,u) = 1 + \int_0^t \frac{\partial}{\partial x} f(s, \underbrace{\phi(s,u)}^{x})\frac{\partial\phi}{\partial u}(s,u)\,ds$$

Let $\boxed{z(t) = \frac{\partial\phi}{\partial u}(t,u).}$ think of u as fixed

From eq(2), $z(0) = 1$

$$z'(t) = \frac{\partial^2\phi}{\partial t\,\partial u}(t,u) = \frac{\partial f}{\partial x}(t,\phi(t,u))\frac{\partial\phi}{\partial u}(t,u)$$

$$(3)\quad \boxed{z'(t) = \frac{\partial f}{\partial x}(t,\phi(t,u))\; z(t)}$$

From (3) & $z(0) = 1$

From (3) & $z(0)=1$

(4) $$\boxed{z(t) = \exp\left(\int_0^t \frac{\partial f}{\partial x}(s,\phi(s,u))\,ds\right)}$$

(5) $$\boxed{\frac{\partial \phi}{\partial u}(t,u) = \exp\left(\int_0^t \frac{\partial f}{\partial x}(s,\phi(s,u))\,ds\right)}$$

$$p(u) = \phi(1,u)$$

$$\boxed{p'(u) = \exp\left(\int_0^1 \frac{\partial f}{\partial x}(s,\phi(s,u))\,ds\right)}$$

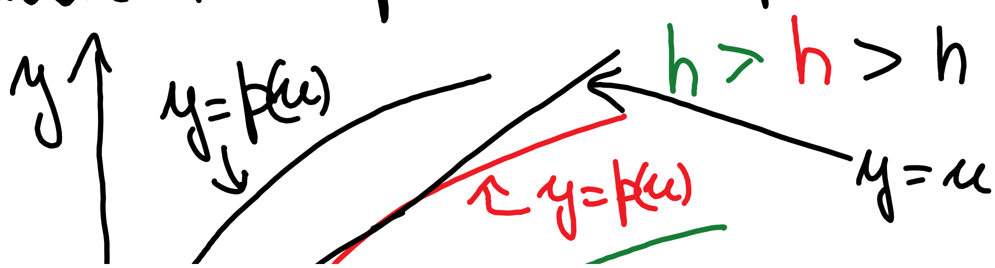
$$p''(u) = p'(u) \int_0^1 \frac{\partial^2 f}{\partial x^2}(s,\phi(s,u))\,\frac{\partial \phi}{\partial u}(s,u)\,ds$$

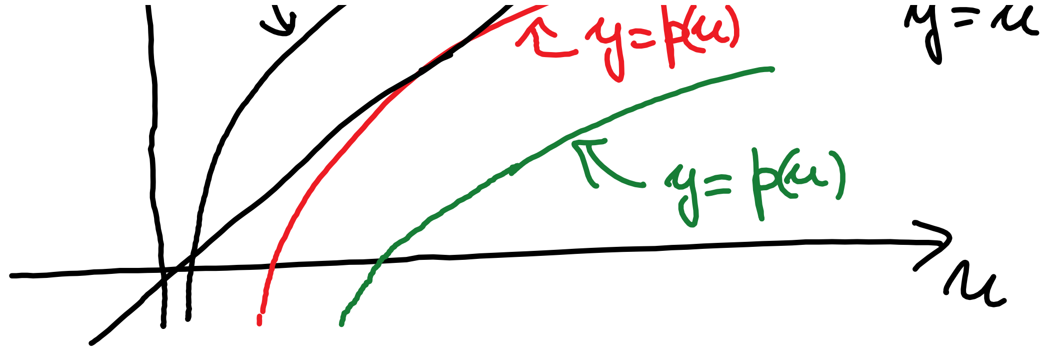
$$\boxed{p''(u) = p'(u) \int_0^1 \frac{\partial^2 f}{\partial x^2}(s,\phi(s,u))\left(e^{\int_0^s \frac{\partial f}{\partial x}(r,\phi(r,u))\,dr}\right)ds}$$

Case $f(t,x) = a\,x(1-x) - h\,(1+\sin(2\pi t))$

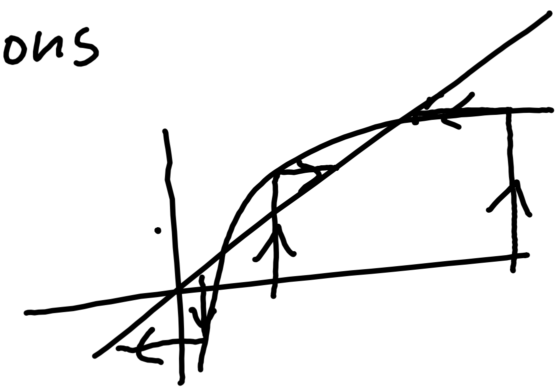
$$\frac{\partial^2 f}{\partial x^2}(t,x) = -2a \qquad a>0$$

We conclude that $p'(u)>0$ & $p''(u)<0$





With some more calculations we can show that there exists h^* such that, if $h<h^*$, there are two 1-periodic solutions. If $h=h^*$, there is only 1-periodic solution. If $h>h^*$, there are no periodic solutions



Def: Let $g:\mathbb{R}\to\mathbb{R}$. We say that u is a fixed point of g if $g(u)=u$

Def: Systems of differential equa

tions

$$x_1' = f_1(t, x_1, x_2, \dots, x_n)$$
$$x_2' = f_2(t, x_1, x_2, \dots, x_n)$$
$$\vdots \qquad\qquad \vdots$$
$$x_n' = f_n(t, x_1, x_2, \dots \dots, x_n)$$

$f_1, f_2, \dots, f_n$ given. t is the independent variable $x' = \frac{dx}{dt}$

<u>Goal</u>: Find the solutions $x_1, x_2, \dots, x_n$

<u>Vector notation</u> $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $x' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$

$$f(t,x) = \begin{bmatrix} f_1(t, x_1, x_2, \dots, x_n) \\ f_2(t, x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(t, x_1, x_2, \dots, x_n) \end{bmatrix}$$

$$x' = f(t, x)$$

Autonomous if $f(t,x) = f(x)$

Autonomous if $f(t,x) = f(x)$

$$x_1' = 2x_1 + x_2^2 \qquad \text{autonomous}$$
$$x_2' = \sin(x_2)\, e^{x_1}$$

$$f(x) = \begin{bmatrix} 2x_1 + x_2^2 \\ \sin(x_2)\, e^{x_1} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Th: $f(t,x) = \begin{bmatrix} f_1(t, x_1, x_2) \\ f_2(t, x_1, x_2) \end{bmatrix}$ $\quad n = 2 \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Let $u \in \mathbb{R}^2$. There exists a unique solution to IVP $\quad x' = f(t,x)$ (in general)

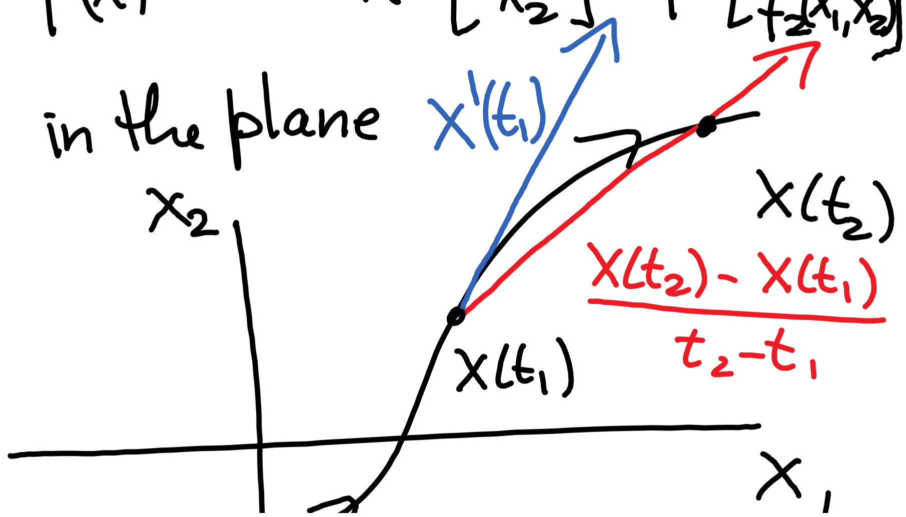
$$x(0) = u$$

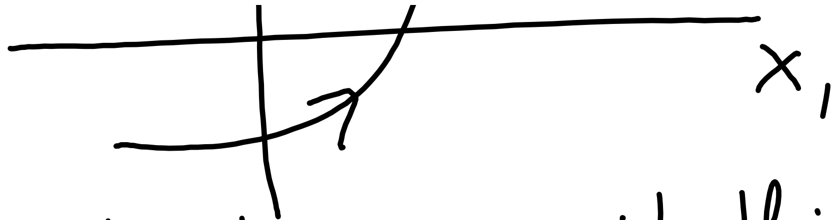
$\boxed{2.2}$ Autonomous systems in $\mathbb{R}^2$

$$x' = f(x) \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad f = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

Recall: Curves in the plane

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$





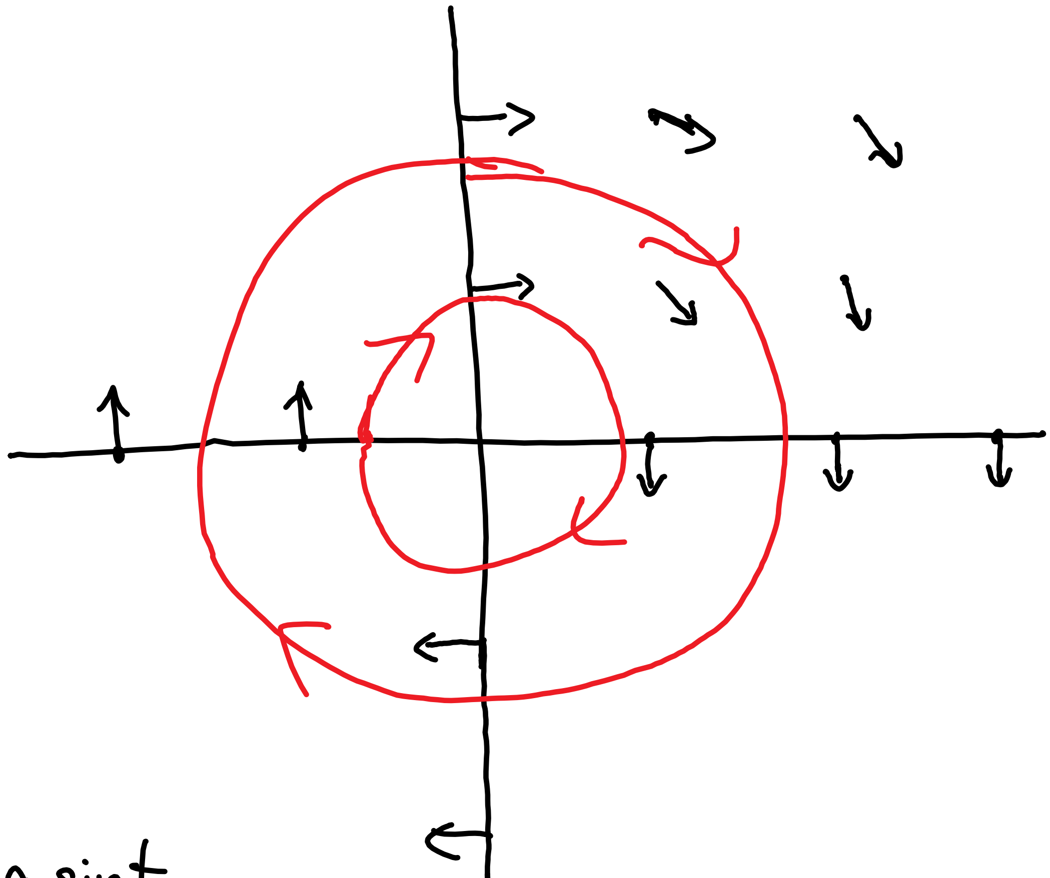
We do not see t when we graph this way.

$x'(t)$ is a vector tangent to the curve $x(t)$.

<u>Direction field</u> Draw a small arrow in the direction of $f(x)$ at the point x. Do this for several different x. The solutions of $x' = f(x)$ are curves that are tangent to these little arrows.

<u>Ex</u>: $x_1' = x_2$ $\quad\quad f(x) = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$

$x_2' = -x_1$



$x_1 = a \sin t$

$x_2 = a \cos t$

$x_1' = a \cos t = x_2$

$x_2' = -a \sin t = -x_1$

then $x_1 = a \sin t$ $\quad x_2 = a \cos t$ is a solution

Linear algebra

$\mathbb{R}$ = set of real numbers

x is a 2-vector if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad x_1, x_2 \in \mathbb{R}$

$\mathbb{R}^2$ = is the set of 2-vectors

Matrices: A is a 2-by-2 matrix if

$A = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$ $\quad a_{ij} \in \mathbb{R}$.

$a_{21} \quad a_{22}]$

$\mathbb{R}^{2\times 2}$ = is the set of 2-by-2 matrices.

<u>Def</u>: $A \in \mathbb{R}^{2\times 2}$. $v \in \mathbb{R}^2$ is an eigenvector of A with eigenvalue $\lambda \in \mathbb{R}$ if

$v \neq 0$ and $Av = \lambda v$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ = identity matrix

$A \in \mathbb{R}^{2\times 2}$ then $\det A = a_{11} a_{22} - a_{21} a_{12}$

<u>Ex</u>: $\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1(3) + 2(2) \\ -1(3) + 0(2) \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$

$\det \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = 1(0) - (-1)2 = 2$

<u>Def</u>: Let $A \in \mathbb{R}^{2\times 2}$.

$P(\lambda) = \det(A - \lambda I)$ = characteristic polynomial

$P(\lambda)$ is a polynomial of degree 2.

λ is an eigenvalue of A if and only if $P(\lambda) = 0$

<u>Steps</u>: 1) Construct $P(\lambda) = \det(A - \lambda I)$

2) Solve $P(\lambda)=0$. these are the eigenvalues

3) For each eigenvalue λ, solve

$(A-\lambda I)v=0$. these are the eigenvectors.

(need $v\neq 0$).

Ex: $A=\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$P(\lambda)=\det(A-\lambda I)=\det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}=$

$(2-\lambda)^2-1$

$P(\lambda)=0 \qquad (2-\lambda)^2-1=0 \qquad \lambda=1,3$

$\lambda_1=1 \qquad \lambda_2=3$

$\boxed{\lambda_1=1}$ $\quad A-\lambda_1 I=\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$v=\begin{bmatrix} 1 \\ -1 \end{bmatrix}t \quad$ set $t=1$ $\quad v=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\boxed{\lambda_2=3}$ $\quad A-\lambda_2 I=\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix}=\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

$v=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigenvalue	Eigenvector
1	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
3	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$