$\boxed{1.4}$ $x' = a\,x(1-x) - h(1+\sin(2\pi t))$

$x' = f(t,x)$

note $f(t+1,x) = f(t,x)$

<u>Obs</u>: Since $f(t+1,x) = f(t,x)$, if $x(t)$ is a solution, so is $x(t+n)$ for all integer n.

<u>Def</u>: $\phi(t,u) = x(t)$ where x is the solution of $x' = f(t,x)$
$x(0) = u$

Equivalently $\frac{\partial \phi}{\partial t} = f(t,\phi)$
$\phi(0,u) = u$

<u>Def</u>: $p : \mathbb{R} \to \mathbb{R}$
$p(u) = \phi(1,u)$

$p(u) = x(1)$, where x is the solution of

$$x' = f(t,x)$$
$$x(0) = u$$

p is called the Poincare map.

<u>Claim</u>: $\phi(t, p(u)) = \phi(t+1, u)$

<u>Proof</u> $x_1(t) = \phi(t, p(u))$
$x_2(t) = \phi(t+1, u)$ are both solutions

of $x' = f(t,x)$ with the same initial

condition $x(0) = u$. then, they are the

same for all t.

<u>Example</u>: 1) $x' = a x$

$$\phi(t, u) = u e^{at}$$
$$p(u) = u e^{a}$$

2) $x' = -x + \cos(t)$

$$x = A \cos(t + \varphi)$$

$$-A\sin(t+\varphi) = -A\cos(t+\varphi) + \cos t$$

$$-A\sin(t)\cos\varphi - A\cos(t)\sin\varphi = -A\cos t\cos\varphi$$
$$+A\sin(t)\sin(\varphi) + \cos(t)$$

$$-A\cos\varphi = A\sin\varphi \qquad \tan\varphi = -1$$
$$-A\sin\varphi = -A\cos\varphi + 1 \qquad \varphi = \frac{3\pi}{4}$$
$$-A = A + 1 \qquad A = -\frac{1}{2}$$

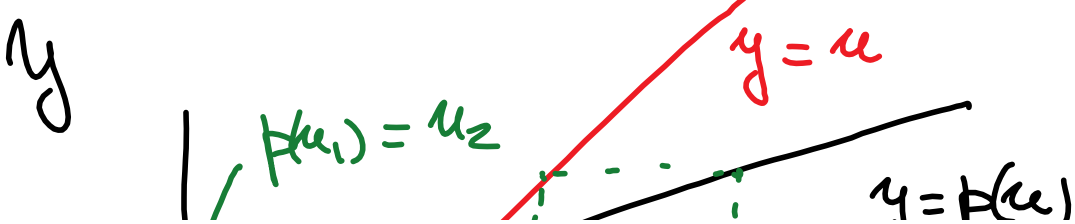
$$x = -\frac{1}{2}\cos\left(t + \frac{3\pi}{4}\right) + k e^{-t} \qquad t=0 \quad x=u$$

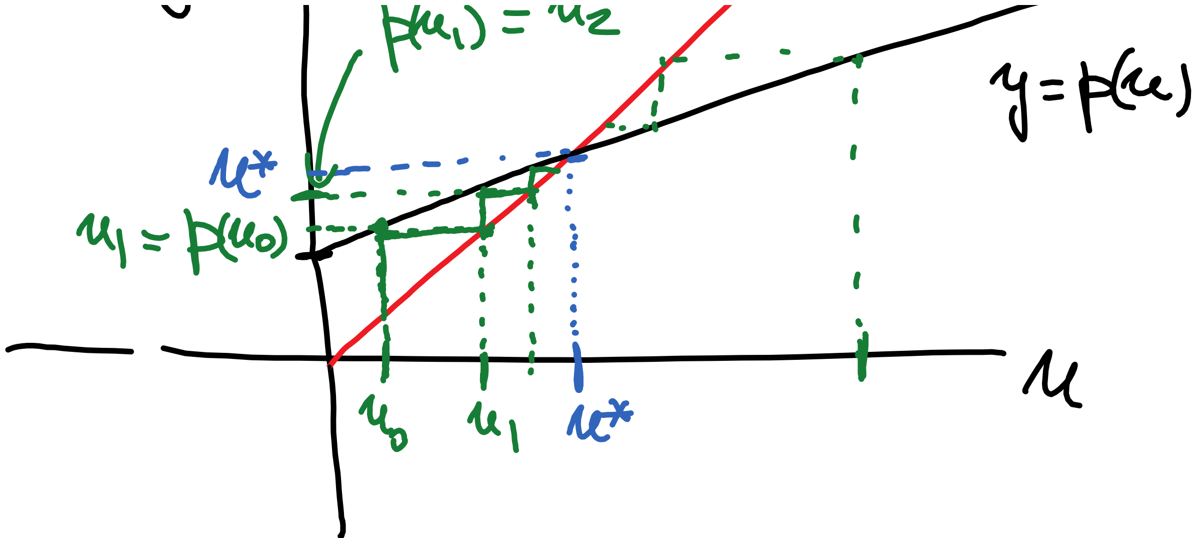
$$u = -\frac{1}{2}\left(-\frac{\sqrt{2}}{2}\right) + k \qquad k = u - \frac{\sqrt{2}}{4}$$

$$\phi(t,u) = x(t) = -\frac{1}{2}\cos\left(t + \frac{3\pi}{4}\right) + \left(u - \frac{\sqrt{2}}{4}\right)e^{-t}$$

$$p(u) = \phi(2\pi, u) = +\frac{\sqrt{2}}{4} + \left(u - \frac{\sqrt{2}}{4}\right)e^{-2\pi} =$$

$$p(u) = \frac{\sqrt{2}}{4}\left(1 - e^{-2\pi}\right) + e^{-2\pi}u$$





$p(u^*) = u^* \qquad \phi(2\pi, u^*) = u^*$

$$u^* = \frac{\sqrt{2}}{4}(1 - e^{-2\pi}) + e^{-2\pi} u^*$$

$$u^* = \frac{\sqrt{2}}{4}$$

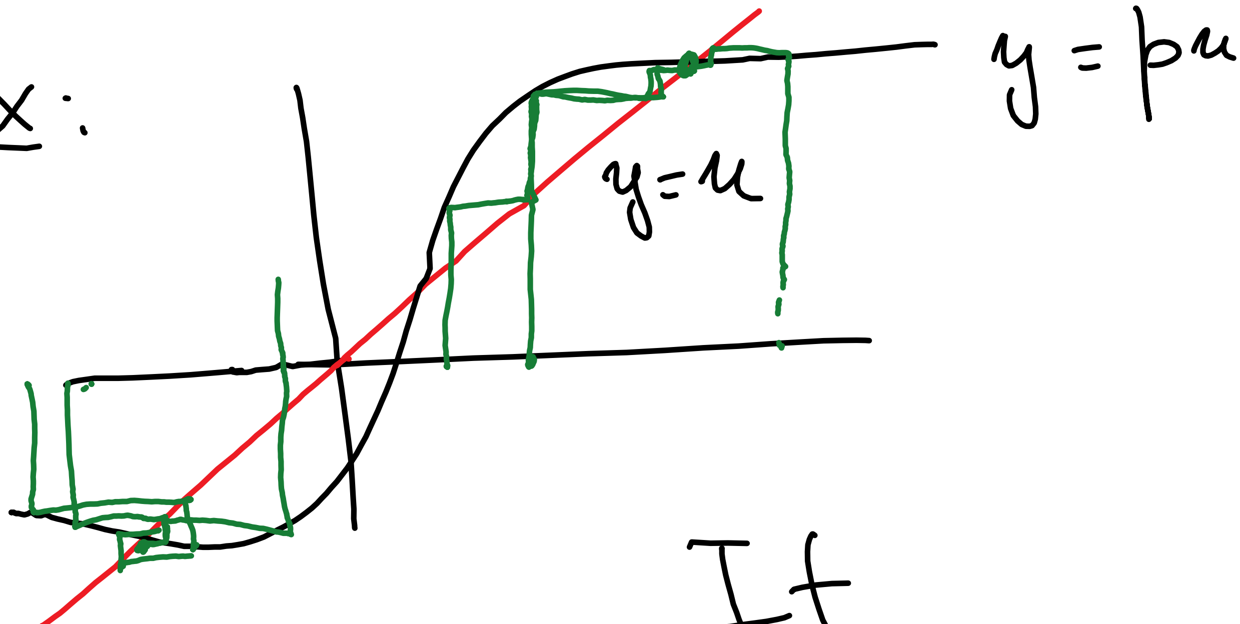
<u>Obs</u>: No matter the initial condition, the solution apprpaches the periodic solution.

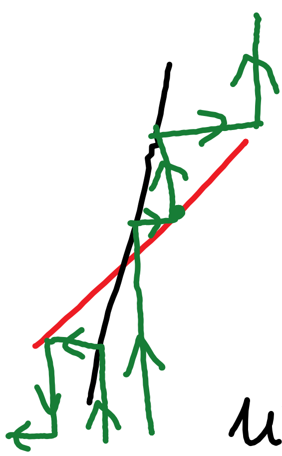
Back to f 1-periodic in t

<u>Obs</u>: If $x' = f(t, u)$
$x(0) = u$

then $x(n) = p^n(u)$. Thus, x is 1-periodic if $p(u) = u$.

<u>Ex</u>:





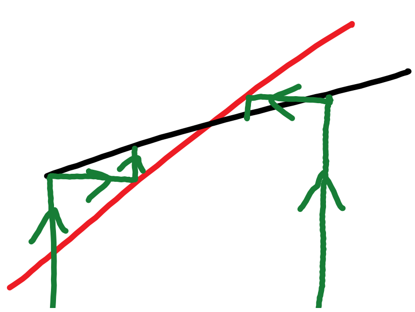
If

$p(u^*) = u^*$

$p'(u^*) > 1$

and u is close to u^*, $x(t)$ goes away from the periodic solution $x' = f(t,x)$

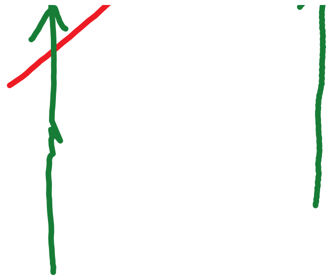
$x(0) = u^*$



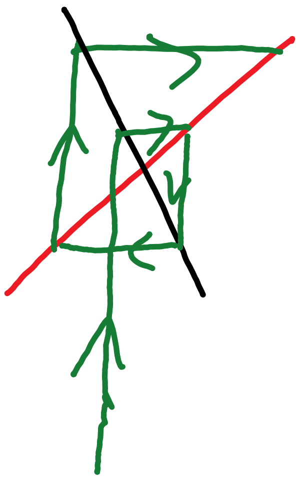
If $p(u^*) = u^*$

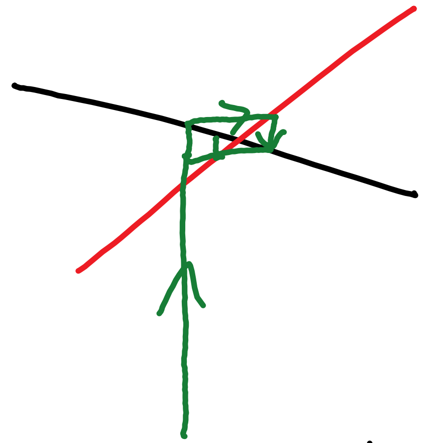
$p'(u^*) = u^*$ then

$x' = f(t,x)$ gets closer



$$\begin{cases} x' = f(t,x) \\ x(0) = u \end{cases}$$ gets closer to the periodic solution





x approaches the periodic trajectory if x starts nearby and $|p'(u^*)| < 1$

x goes away from the periodic solution if $|p'(u^*)| > 1$. Even if we start close to u^*.

$\boxed{1.5}$ Replace t by s to get

$$\frac{\partial}{\partial s}\phi(s,u) = f(s, \phi(s,u))$$

integrate over s from 0 to t

$$(1)\quad \phi(t,u) = u + \int_0^t f(s, \phi(s,u))\,ds$$

Take $\frac{\partial}{\partial u}$ of eq(1)

$$(2)\quad \frac{\partial \phi}{\partial u}(t,u) = 1 + \int_0^t \frac{\partial}{\partial x} f(s, \underbrace{\phi(s,u)}_{x})\, \frac{\partial \phi}{\partial u}(s,u)\,ds$$

Let $z(t) = \frac{\partial \phi}{\partial u}(t,u)$. think of u as fixed

From eq(2), $z(0) = 1$

$$z'(t) = \frac{\partial^2 \phi}{\partial t\,\partial u}(t,u) = \frac{\partial f}{\partial x}(t, \phi(t,u))\,\frac{\partial \phi}{\partial u}(t,u)$$