<u>Distances on a set S</u> : This as a function that takes two elemens of S and gives us a non-negative number.

$$d: S \times S \longrightarrow \mathbb{R}_{\geq 0}$$

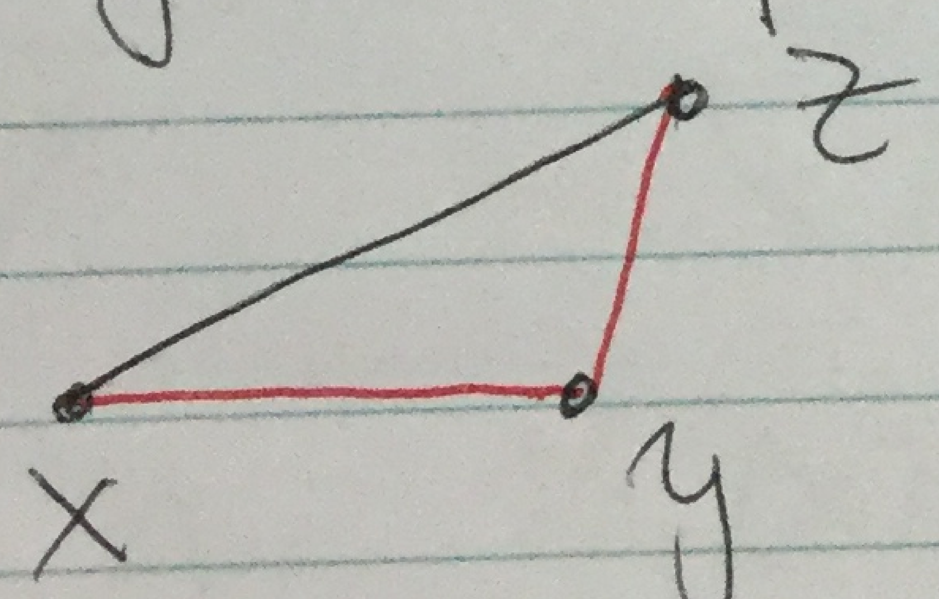
that satisfies

2) $d(x,y) = d(y,x)$

1) $\cancel{d(x,x)}$ $d(x,y) = 0 \iff x = y$

3) $d(x,z) \leq d(x,y) + d(y,z)$

triangle inequality



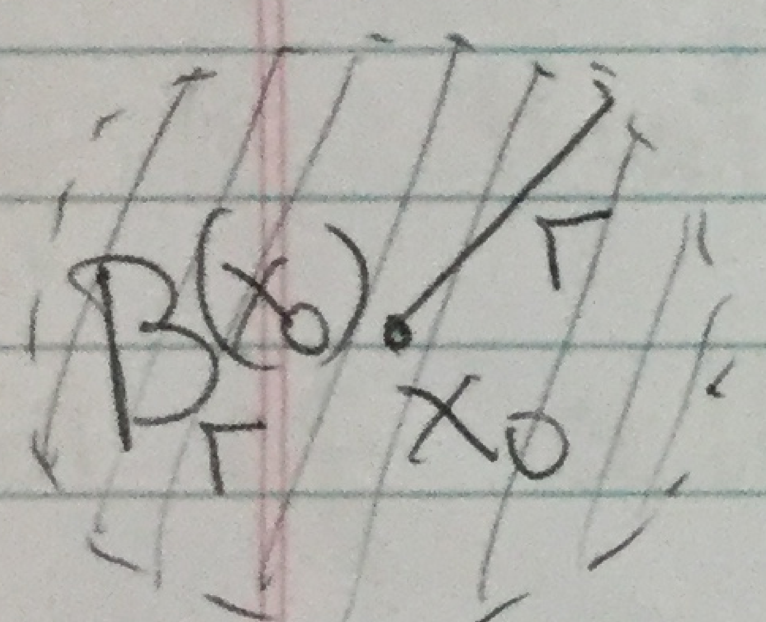
a set with a distance is called a metric space

<u>Example</u> $S = \mathbb{R}^n$ d is the Euclidean

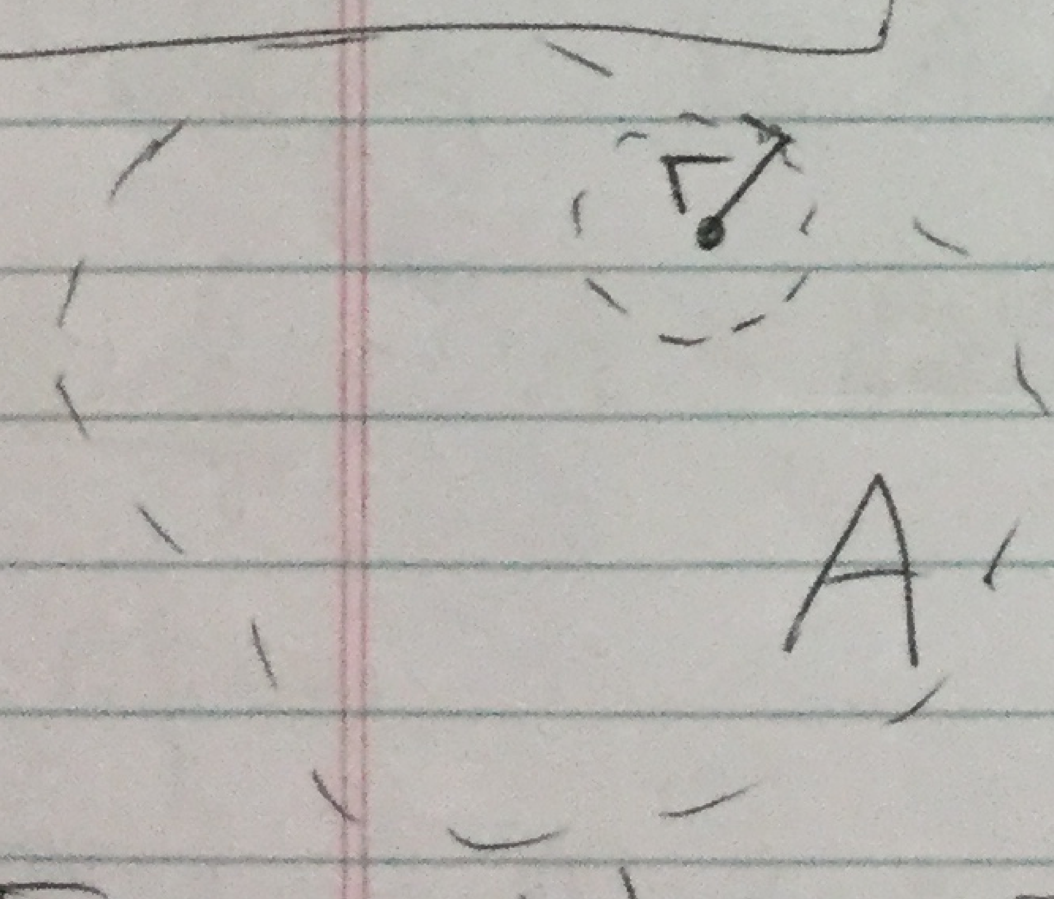
distance $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$$d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2}$$

<u>Def</u>: Let $x_0 \in \mathbb{R}^n$. Let $r > 0$ the ball centered at x_0 with radius r is $B_r(x_0) = \{x \in \mathbb{R}^n : d(x, x_0) < r\}$



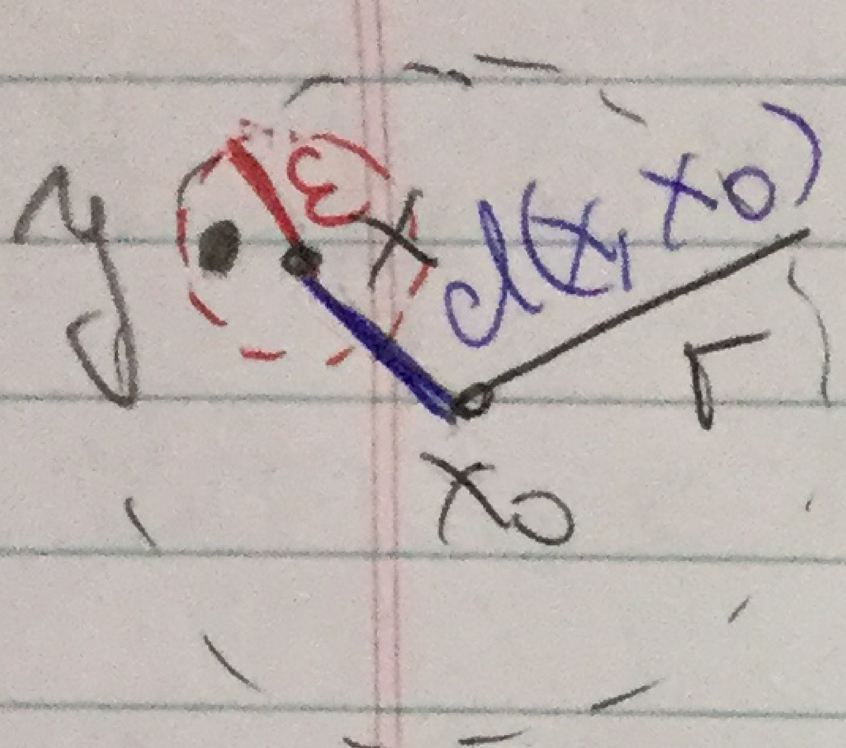
<u>Def</u>: $A \subset \mathbb{R}^n$ is an open set if for all $x \in A$ there exist $r > 0$ (that depends on x and A) such that $B_r(x) \subset A$.



<u>Example</u>: The open balls are open.

<u>proof</u> Let $x_0 \in \mathbb{R}^n$ and $r > 0$. Let $x \in B_r(x_0)$. Let $\varepsilon = r - d(x_0, x)$.

$\varepsilon > 0$ because $x \in B_r(x_0)$



We need to show that $B_\varepsilon(x) \subset B_r(x_0)$. Let $y \in B_\varepsilon(x)$,

then $d(x_0, y) \le d(x_0, x) + d(x, y) < d(x_0, x) + \varepsilon$

$= d(x_0, x) + r - d(x_0, x) = r$ thus

$y \in B_r(x_0)$

<u>Def</u>: Let $A \subset \mathbb{R}^n$. The complement of A, that is denoted by A^c, is the set of all points in $\mathbb{R}^n$ that do not belong to A.

<u>Def</u>: $A \subset \mathbb{R}^n$. A is closed if A^c is open

<u>Def</u>: $A \subset \mathbb{R}^n$ is bounded if there exists $x_0 \in \mathbb{R}^n$ and $r > 0$ such that

$A \subset B_r(x_0)$

Def: $x' = F(x)$ in $\mathbb{R}^n$

Let $x_0 \in \mathbb{R}^n$. Then:

1) $\omega(x_0) = \{ x \in \mathbb{R}^n :$ there exists $t_1 < t_2 < \dots < t_n <$ with $t_n \xrightarrow[n \to \infty]{} \infty$ such that $\phi_{t_n}(x_0) \to x \}$

$\omega(x_0)$ is called the ω-limit set through x_0

S is an ω-limit set if $S = \omega(x_0)$ for some x_0

2) $\alpha(x_0) = \{ x \in \mathbb{R}^n :$ there exists $t_1 > t_2 > t_3 > \dots$ with $t_n \xrightarrow[n \to \infty]{} -\infty$ such that $\phi_{t_n}(x_0) \to x \}$

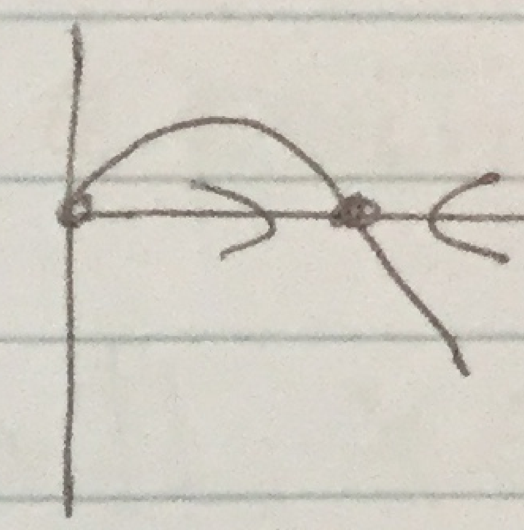
$\alpha(x_0)$ is called the α-limit set through x_0. S is an α-limit set if $S = \alpha(x_0)$ for some x_0.

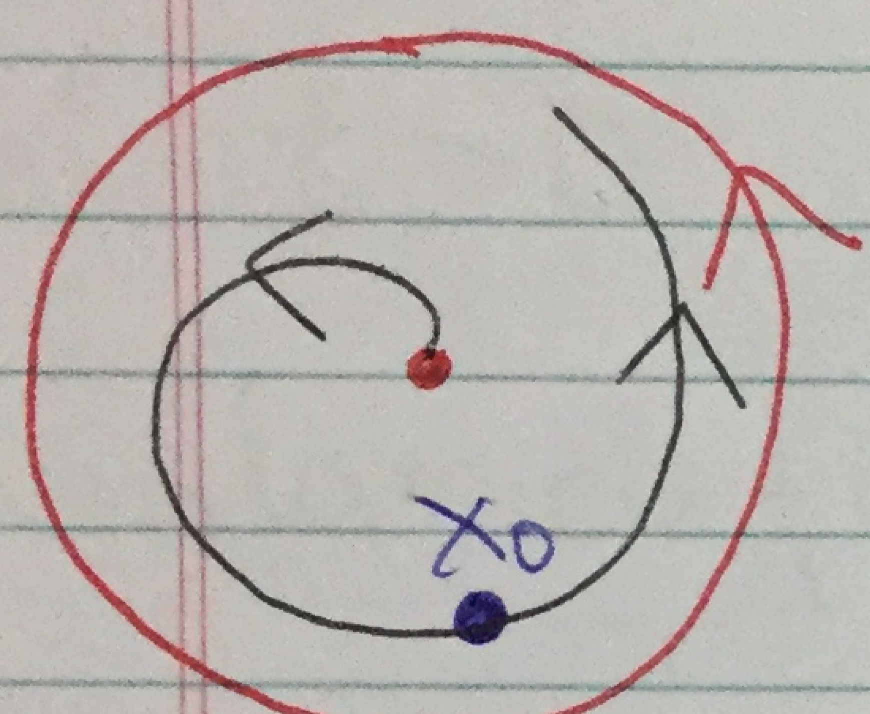
If $x \in \alpha(x_0)$, we say that x is an

α-limit point for ~~the~~ the solution through x_0.

~~Def~~ Ex

$$r' = r - r^2$$
$$\theta' = 1$$

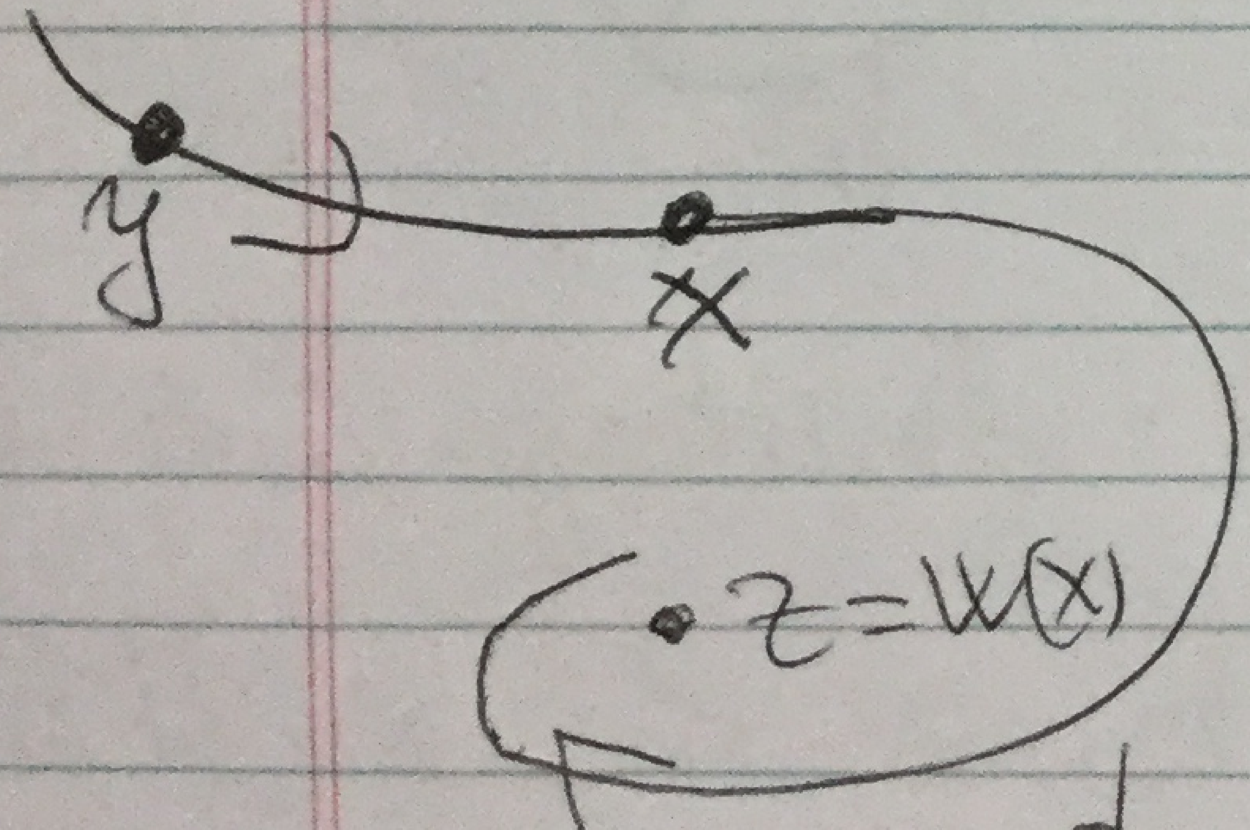




$$\omega(x_0) = \{(x,y) : x^2 + y^2 = 1\}$$
$$\alpha(x_0) = \{(0,0)\} = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$$

$(0,0)$ and $\begin{bmatrix}0\\0\end{bmatrix}$ are the same.

Prop: x and y lie in the same solution. Then $\omega(x) = \omega(y)$

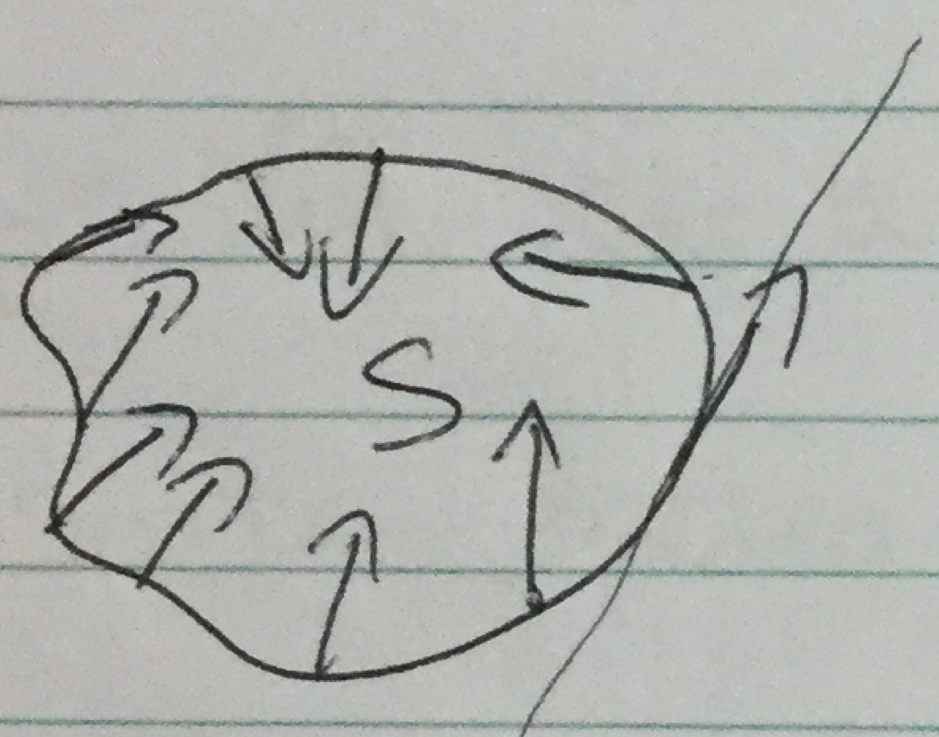


proof: Let $z \in \omega(x)$. Then there exist $t_n \to \infty$ and increasing such that $\phi_{t_n}(x) \to z$. Let τ such that $\phi_\tau(y) = x$. Then $\phi_{t_n}(x) = \phi_{t_n}(\phi_\tau(y)) = \phi_{t_n+\tau}(y)$

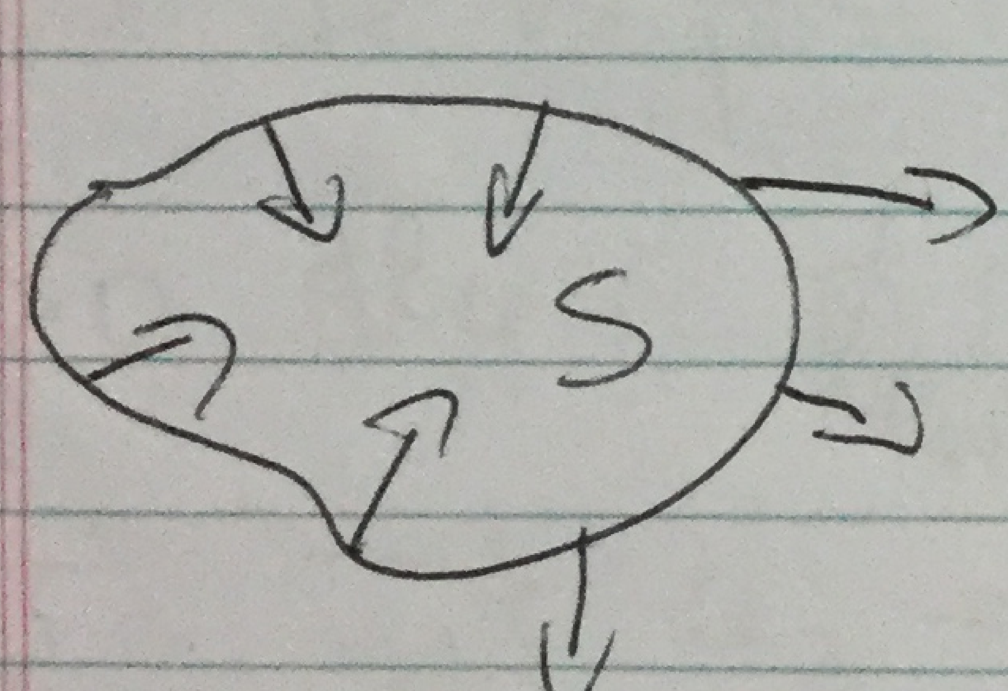
thus, $\phi_{t_n+\delta}(y) \xrightarrow[n\to\infty]{} z$ ✓

Def: $x' = F(x)$. $S \subset \mathbb{R}^n$. S is a positively invariant set if $\phi_t(x) \in S$ for all $x \in S$ and $t > 0$

Similar definition for negatively invariant

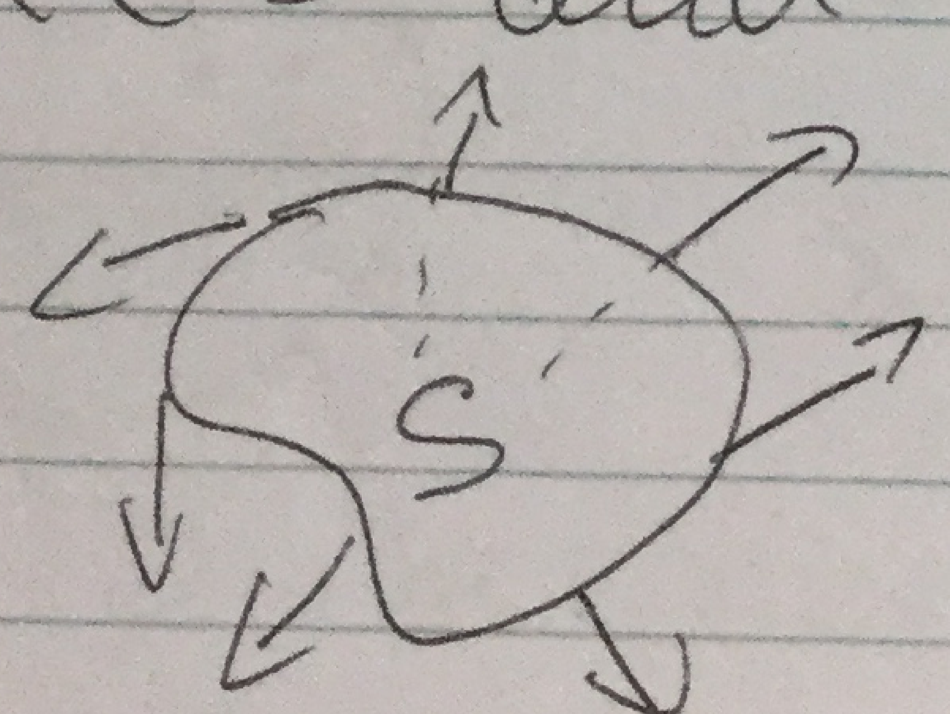


S is positively invariant

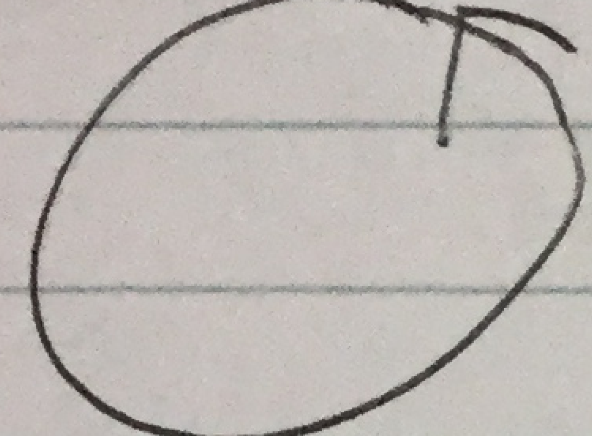


S is not positively invariant

S is negatively invariant if $\phi_t(x) \in S$ for all $x \in S$ and $t < 0$



S is negatively invariant

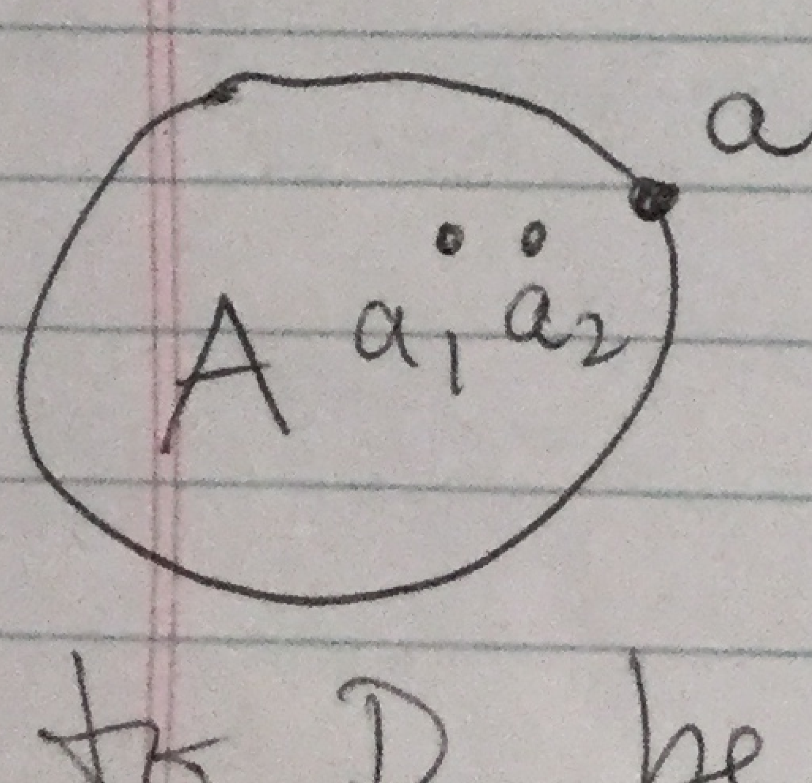
Ex:  A closed trajectory is both positively and negative-ly invariant

Prop: $D \subset \mathbb{R}^n$. D positively invariant. $z \in D$. then $\omega(z) \subset D$

proof: Let $x \in \omega(z)$. then $\phi_{t_n}(z) \xrightarrow[n\to\infty]{} x$ for some increasing sequence t_n that $t_n \to \infty$

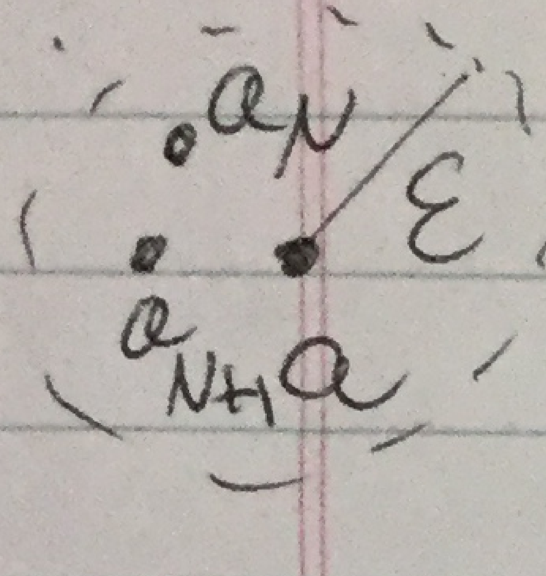
Since D is positively invariant, $z \in D$ and $t_n > 0$ then $\phi_{t_n}(z) \in D$.

Fact: If $a_n \to a$ and $a_n \in A$ and A is closed then $a \in A$ $(A \subset \mathbb{R}^n)$



From this fact, $x = \lim_{n\to\infty} \phi_{t_n}(z)$ also belongs to D because D is closed.

<u>Reminder</u> $a_n \in \mathbb{R}^n$ $a \in \mathbb{R}$. $a_n \to a$

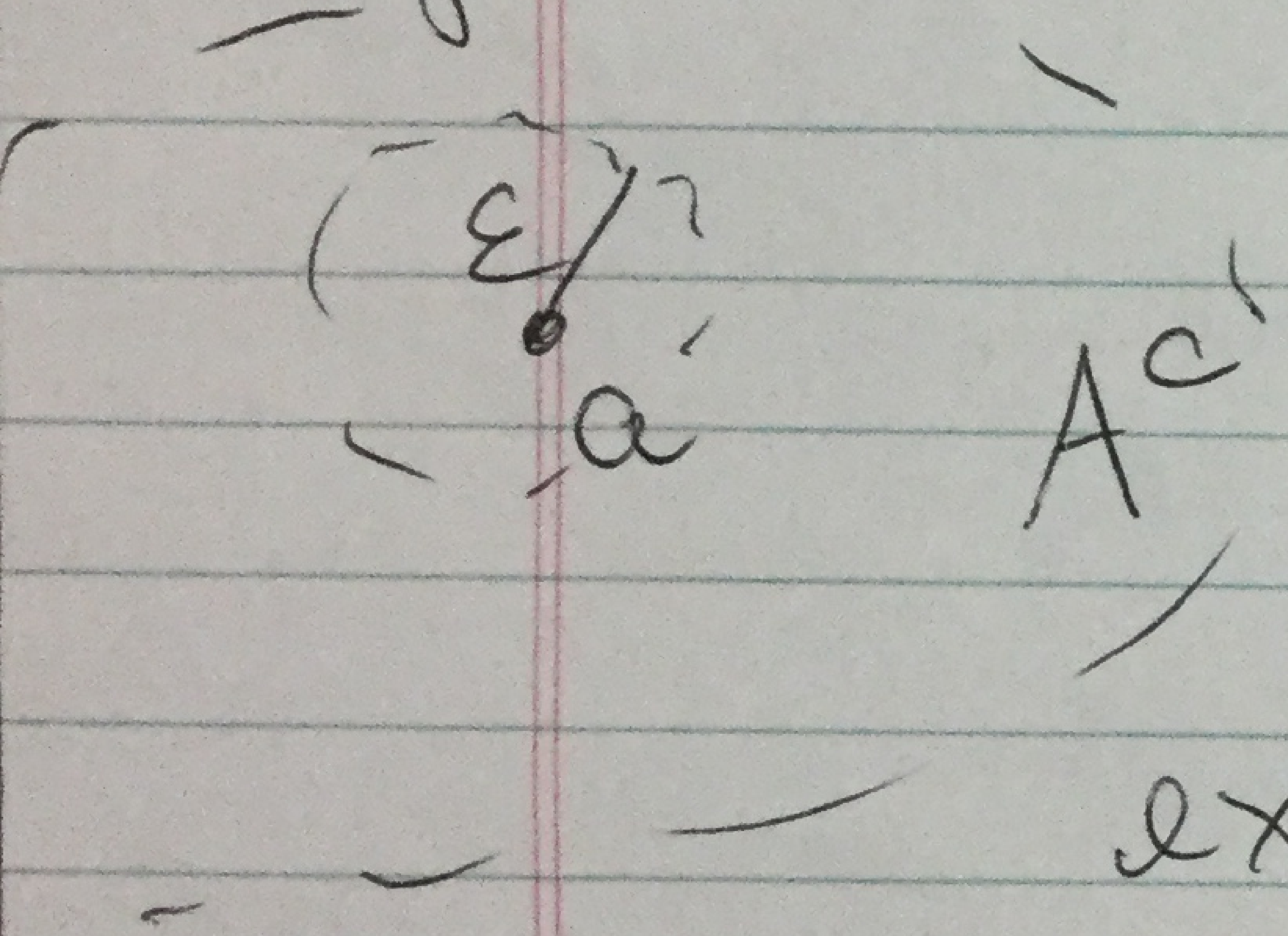


For all $\varepsilon > 0$, $\exists N$ such that $n \geq N$ then $d(a_n, a) < \varepsilon$

<u>proof of Fact</u>: $a_n \to a$ $a_n \in A$ closed.

I want to show that $a \in A$.

<u>By</u> <u>contradiction</u>. If $a \notin A$, then



$a \in A^c$. A closed means A^c open. Thus, there exists $\varepsilon > 0$ such that $B_\varepsilon(a) \subset A^c$. Since $a_n \to a$, $\exists N$ such that $d(a_N, a) < \varepsilon$. That would mean $a_N \notin A$, which is a contradiction