Recall: $x' = F(x)$ $\quad x = x(t) \in \mathbb{R}^n$ ①

$F = F(x) \in \mathbb{R}^n$

$\phi_t(x)$ is the solution $\phi_0(x) = x$

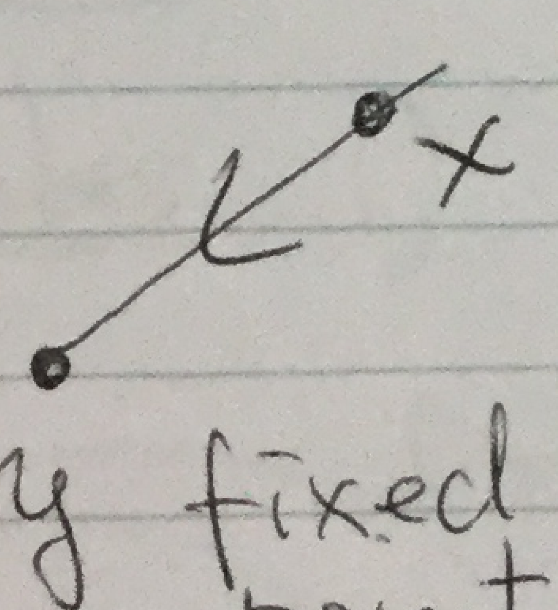
$$\frac{d}{dt}\phi_t(x) = F(\phi_t(x))$$

$$\phi_0(x) = x$$

In other words, $\phi_t(x)$ is the solution of $z' = F(z)$ $\quad z(0) = x$

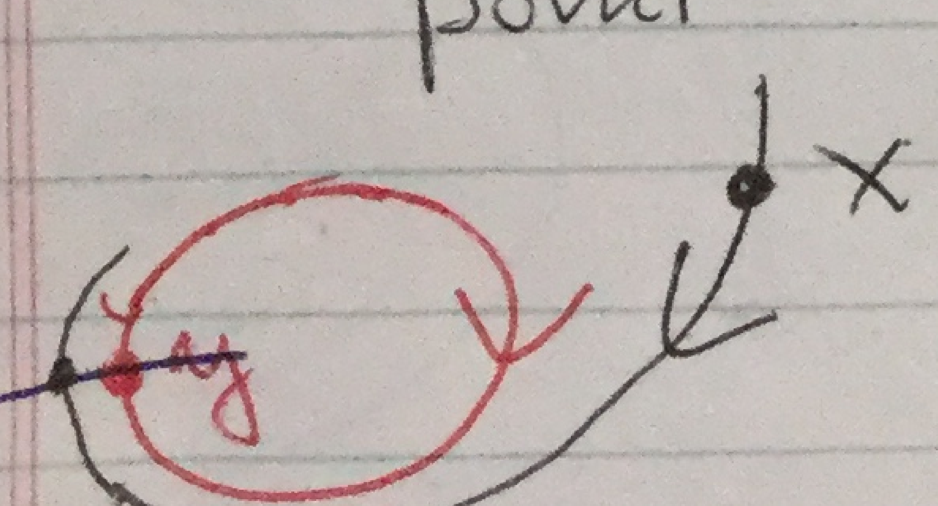
Def: $y \in \mathbb{R}^n$ is an ω-limit point for the solution through x if there exists a sequence $t_n \to +\infty$ such that $\phi_{t_n}(x) \to y$

Ex: 1)



y is an ω-limit point for the solution through x

2)



y is an ω-limit point for the solution through x

3) x is an ω-limit point for the solution through x.

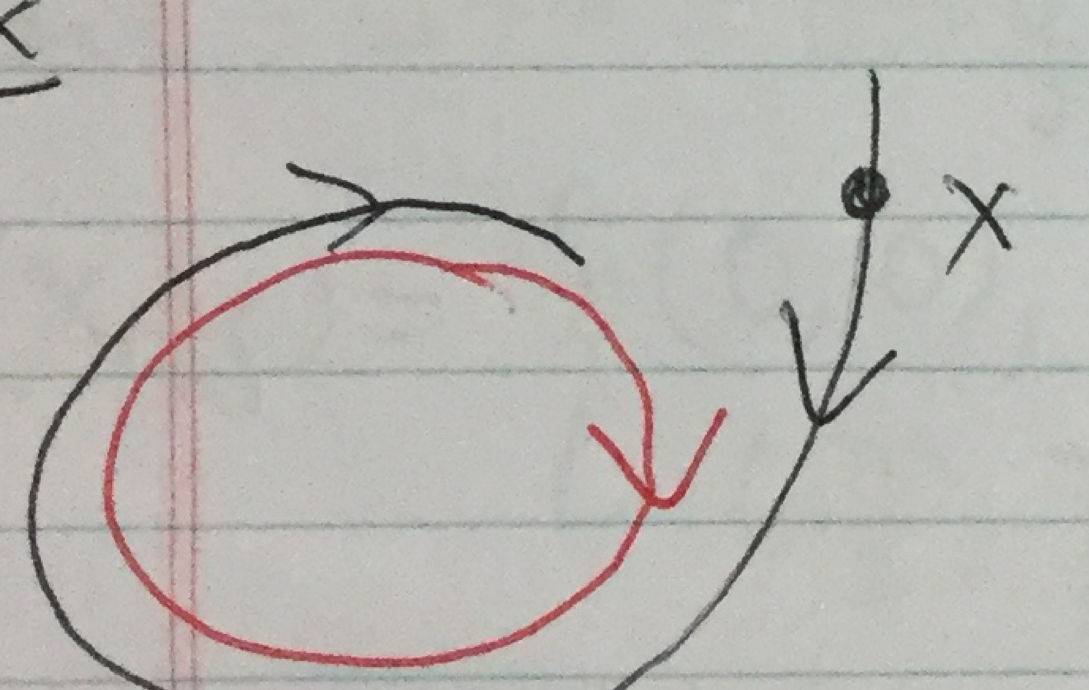
$\dot{}$ x
fixed point

Def: $\omega(x) = \{y \in \mathbb{R}^n : y$ is an ω-limit point for the solution through $x\}$

$\omega(x)$ is called the ω-limit set of x

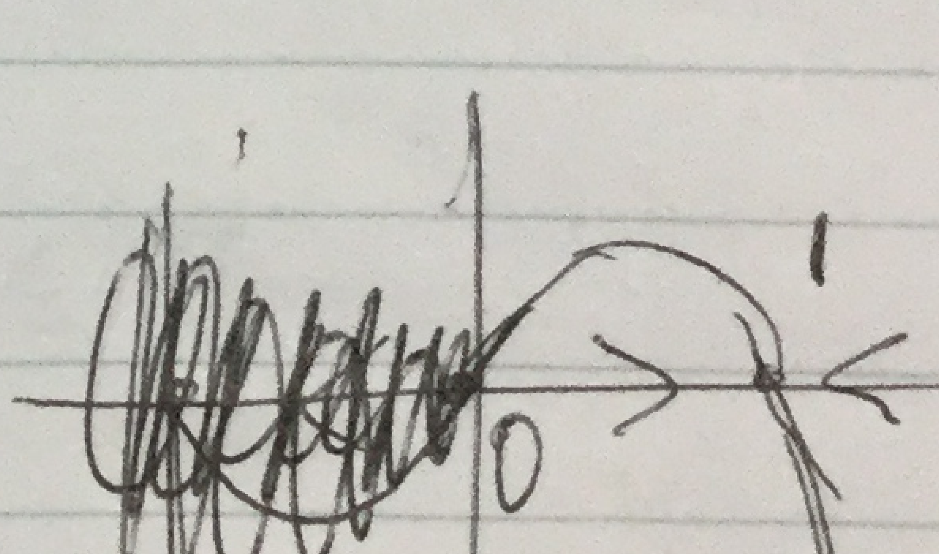
We say that S is an ω-limit set if $S = \omega(x)$ for some x.

Ex

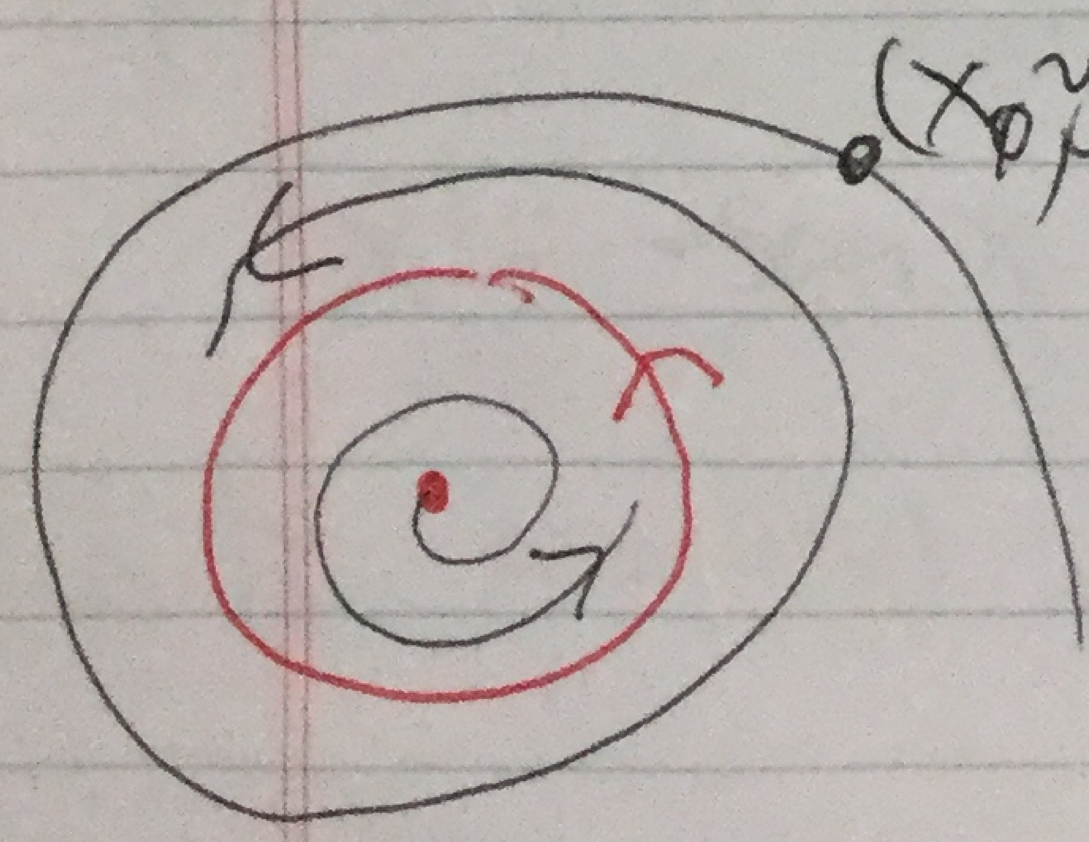


Red cycle (points in the red solution) $= \omega(x)$

Examples 1) $r' = r - r^3$



$\theta' = 1$



If $(x_0, y_0) \neq (0,0)$ then

$\omega(x_0, y_0) = \{(x,y) : x^2 + y^2 = 1\}$

<u>Ex</u>: $x' = \sin x(-0.1\cos x - \cos y)$

$$y' = \sin y(\cos x - 0.1\cos y)$$

Fixed points $\sin x(-0.1\cos x - \cos y) = 0$
$\sin y(\cos x - 0.1\cos y) = 0$

$\sin x = 0$ or $-0.1\cos x - \cos y = 0$

If $-0.1\cos x - \cos y = 0$ and $\sin y = 0$

then $\cos y = \pm 1$ then $\cos x = \pm 10$ impossible

If $-0.1\cos x - \cos y = 0$ then $\cos x - 0.1\cos y = 0$

then $\cos x = \cos y = 0$ or $\sin x = \sin y = 0$

$$(x, y) = \left\{ \begin{array}{l} (0,0), (0,\pi), (\pi, 0), (\pi, \pi) \\ (\frac{\pi}{2}, \frac{\pi}{2}) \end{array} \right.$$

focus on $0 \leq x \leq \pi$ & $0 \leq y \leq \pi$

Replace 0.1 by 0 to get

$$x' = -\sin x \cos y \qquad y' = \sin y \cos x$$

think $y = y(x)$ then $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\sin y \cos x}{-\sin x \cos y}$

$$\frac{dy}{dx} = -\frac{\tan y}{\tan x} \qquad dy \cot y = -dx \cot x$$

$$\ln|\sin y| = -\ln|\sin x| + C$$

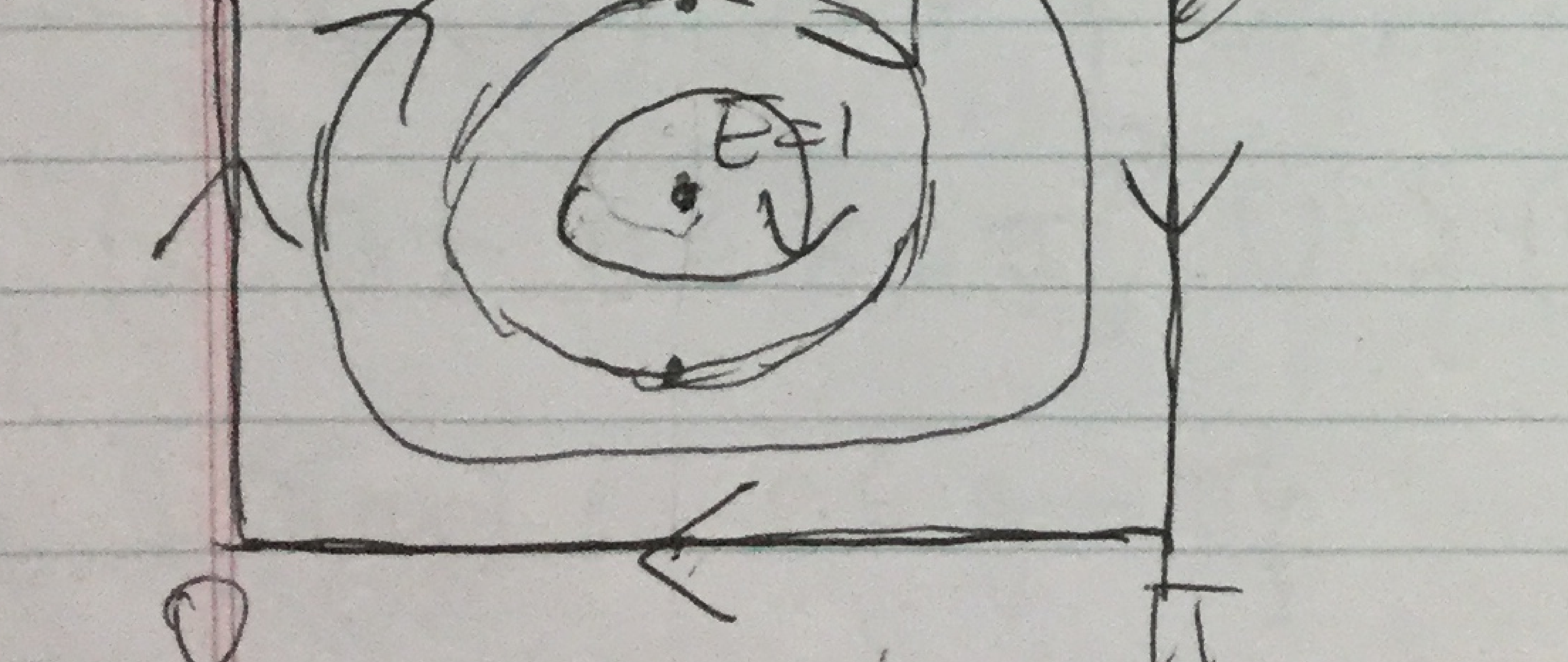
$$|\sin y| = \frac{e^C}{|\sin x|} \qquad \sin x \overset{\sin}{\cancel{\cos}} y = K$$

$$E(x,y) = \sin x \sin y$$

$$\frac{d}{dt}E = \frac{\partial E}{\partial x}x' + \frac{\partial E}{\partial y}y' = \cos x \sin y(-\sin x \cos y) +$$

$$+ \sin x \cos y (\sin y \cos x) = 0$$

Phase plane



level sets of

$E = \sin x \sin y$

$x' = -\sin x \cos \cancel{x} y$

$y' = \sin y \cos x$

Back to original system

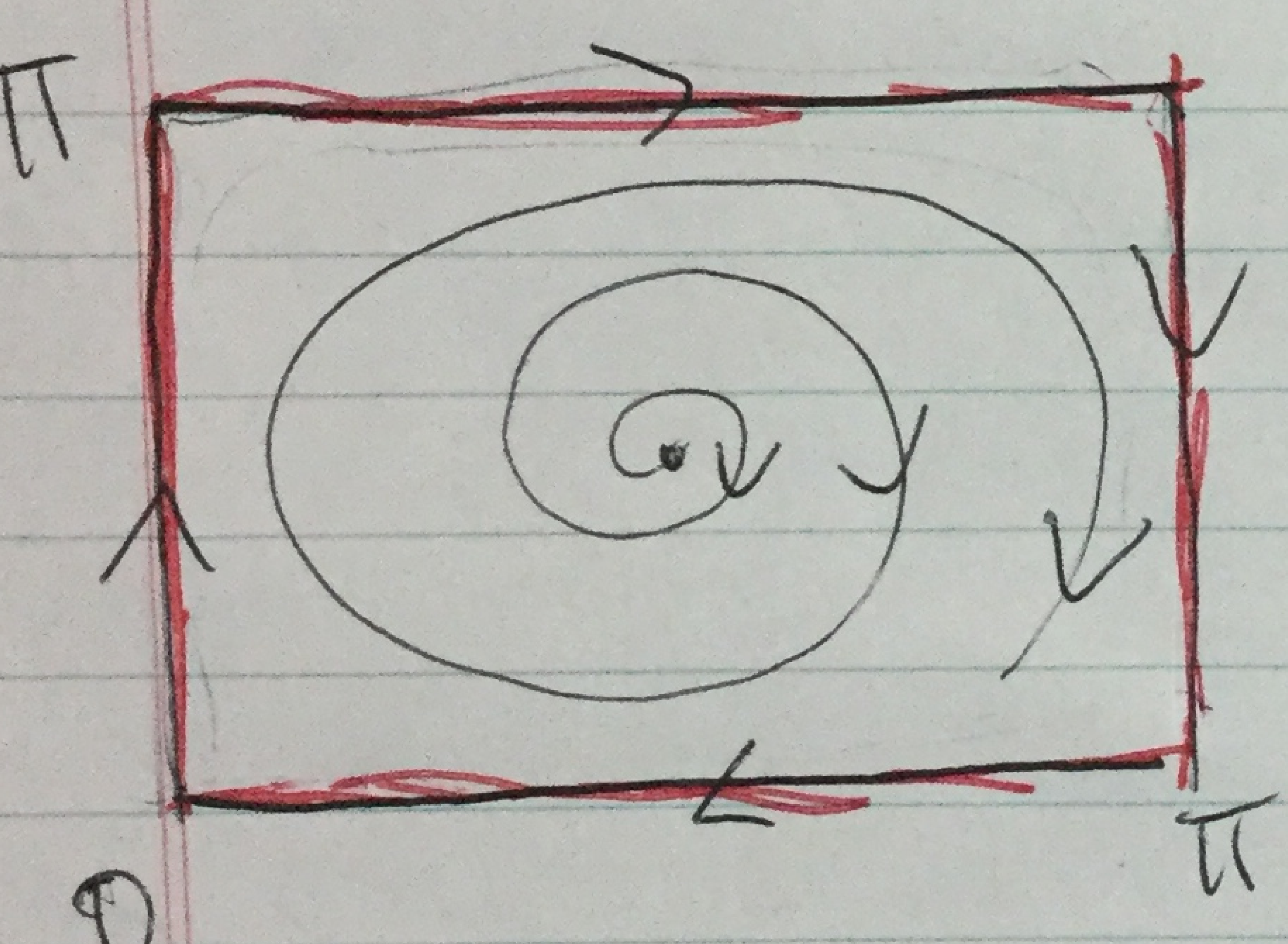
$$x' = \sin x(-0.1 \cos x - \cos y)$$
$$y' = \sin y(\cos x - 0.1 \cos y)$$

$E = \sin x \sin y$

$\frac{dE}{dt} = \frac{\partial E}{\partial x} x' + \frac{\partial E}{\partial y} y' = \cos x \sin y \sin x (-0.1$

$\cos x - \cos y) + \sin x \cos y \sin y (\cos x - 0.1 \cos y) =$

$= -0.1 \sin x \sin y (\cos^2 x + \cos^2 y) < 0$



for any $(x_0, y_0) \neq \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ $0 < x_0 < \pi$ & $0 < y_0 < \pi$

then $\omega(x_0, y_0) = \{(x, 0) : 0 \leq x \leq \pi\} \cup$

$\{(x, \pi) : 0 \leq x \leq \pi\} \cup \{(0, y) : 0 \leq y \leq \pi\} \cup$

$\cup \{(\pi, y) : 0 \leq y \leq \pi\}$

the red lines