Example:

$\theta' = v$ ← angular velocity

$v' = -\sin\theta - bv + k$

↑ gravity ↑ damping ↑ applied torque

$k \geq 0 \quad b \geq 0$

i) Find all equilibrium points

$v = 0 \qquad -\sin\theta - bv + k = 0$

Case a) $\quad 0 \leq k \leq 1 \qquad \theta^* = \sin^{-1} k$

critical points $\quad \theta = \theta^* + 2\pi n \quad$ or $\quad \theta = \pi - \theta^* + 2\pi n$

$v = 0 \qquad n \in \mathbb{R}$

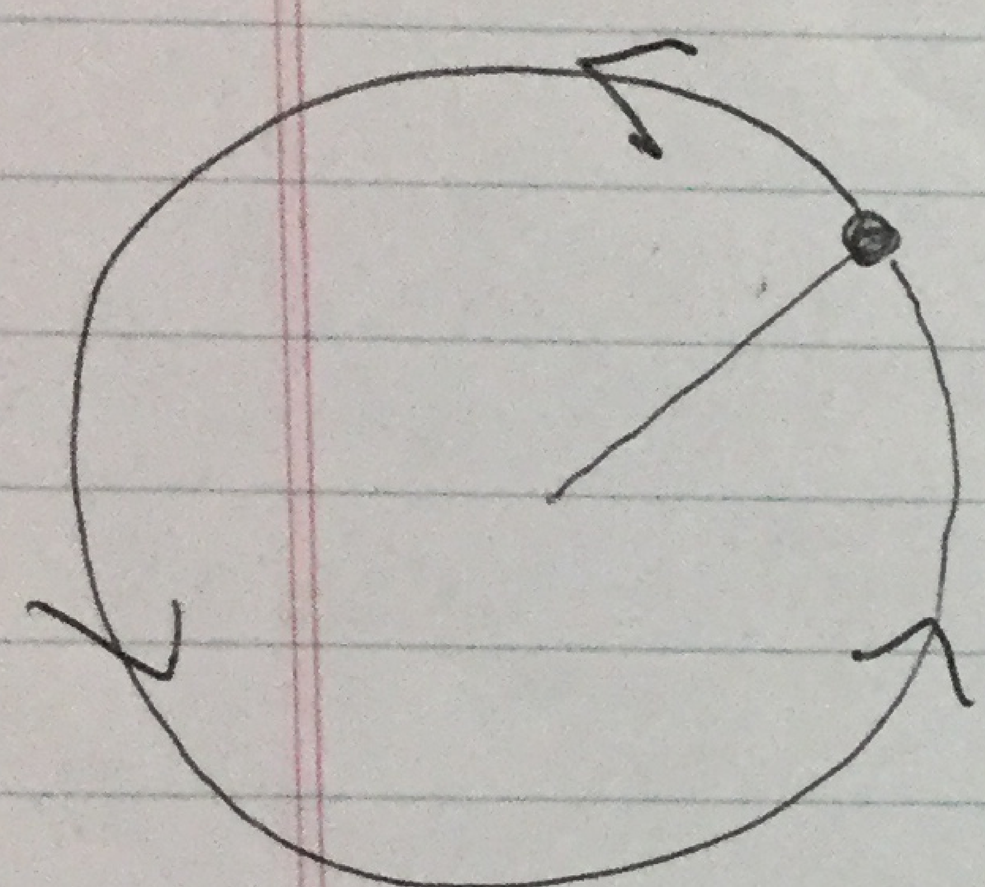
Case b) $\quad k > 1$ no fixed points

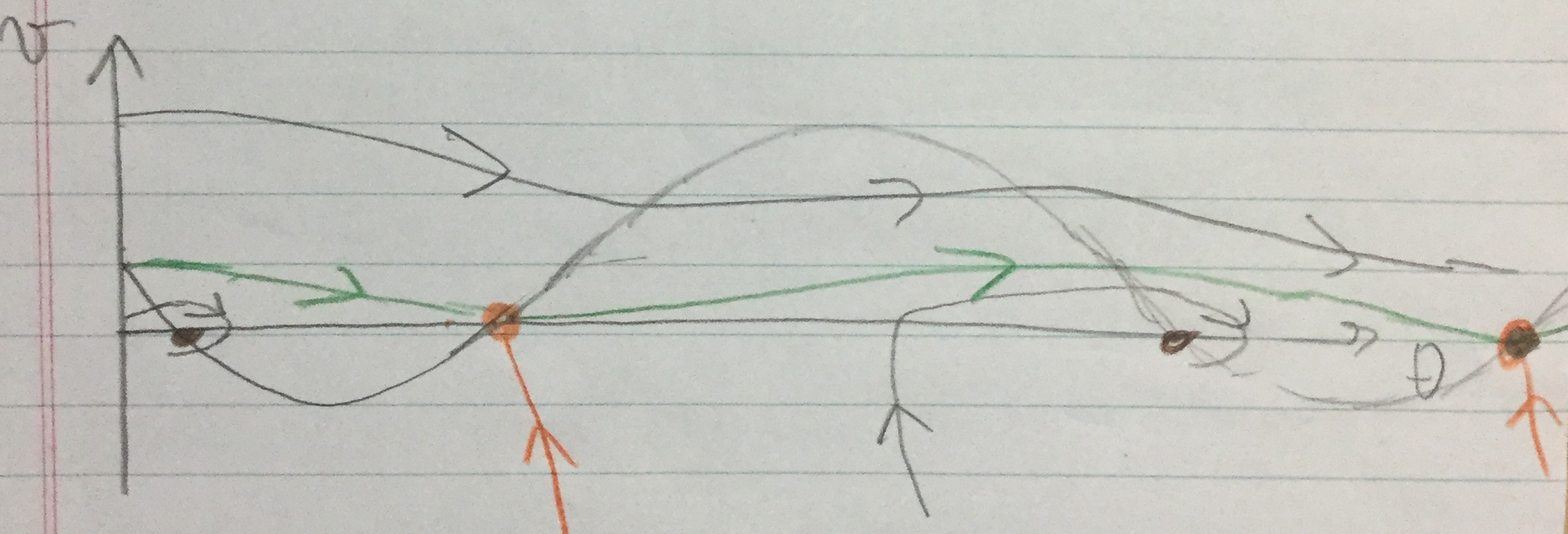
Stability of fixed points

$$A = \begin{bmatrix} \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial v} \\ \frac{\partial(-\sin\theta - bv + k)}{\partial \theta} & \frac{\partial(-\sin\theta - bv + k)}{\partial v} \end{bmatrix}$$

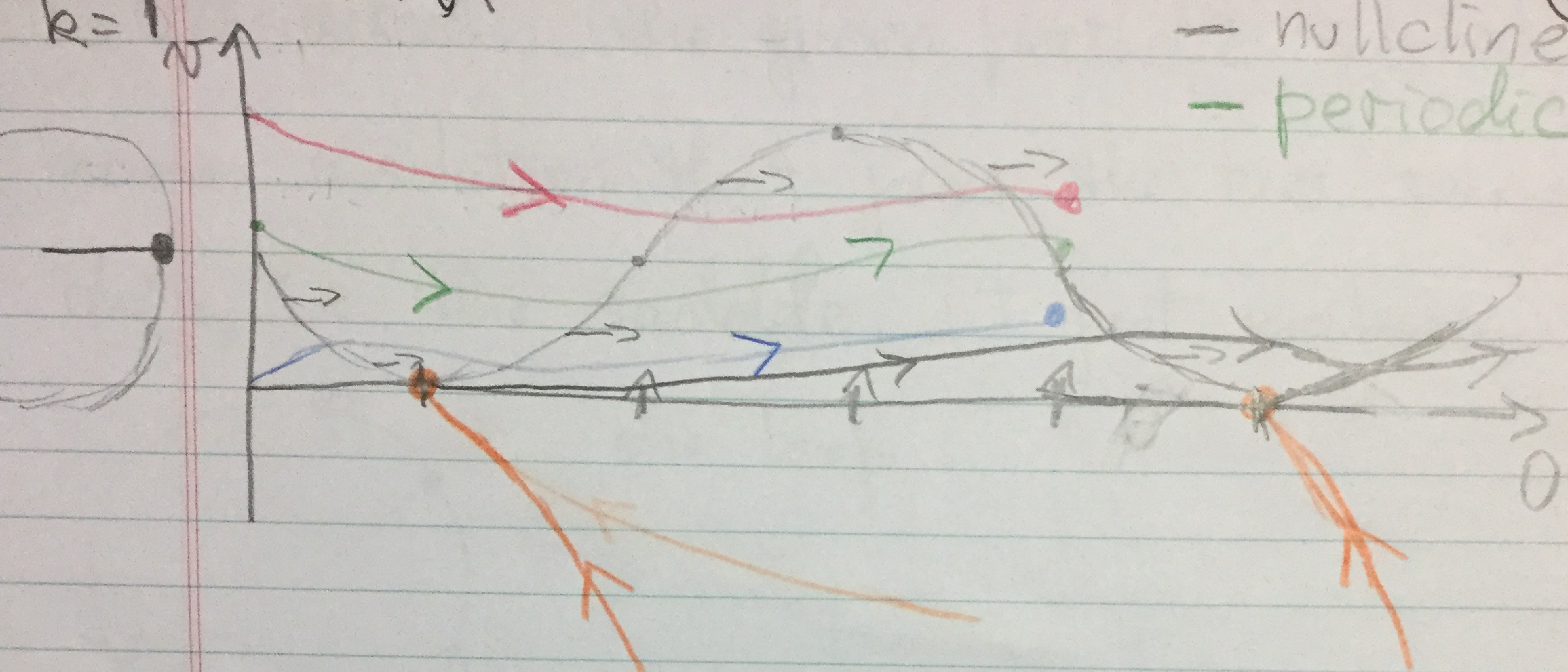
$$A = \begin{bmatrix} 0 & 1 \\ -\cos\theta & -b \end{bmatrix}$$

As k decreases the green periodic solution becomes a solution that joins the two orange dots, i.e. the saddle. It just makes one loop

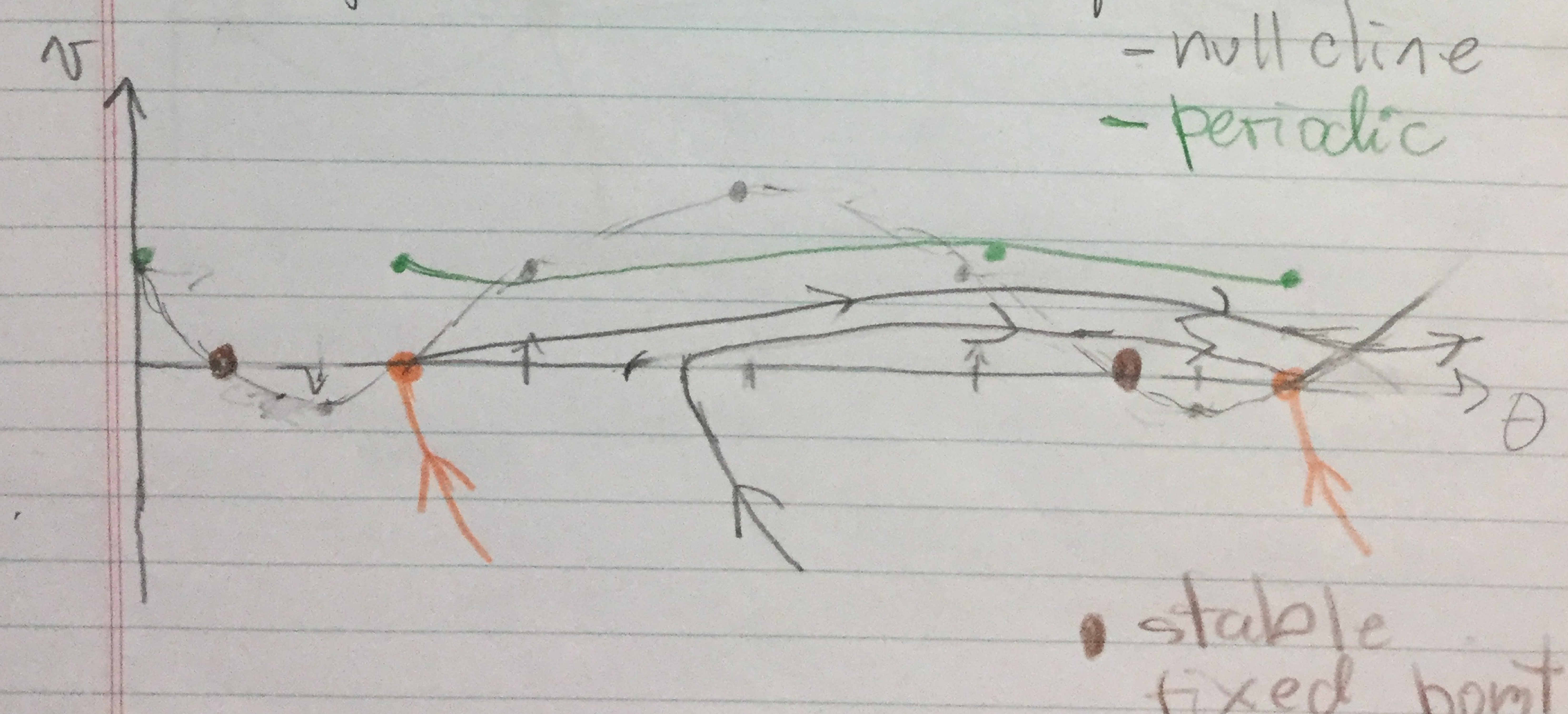




$b>0$ fixed



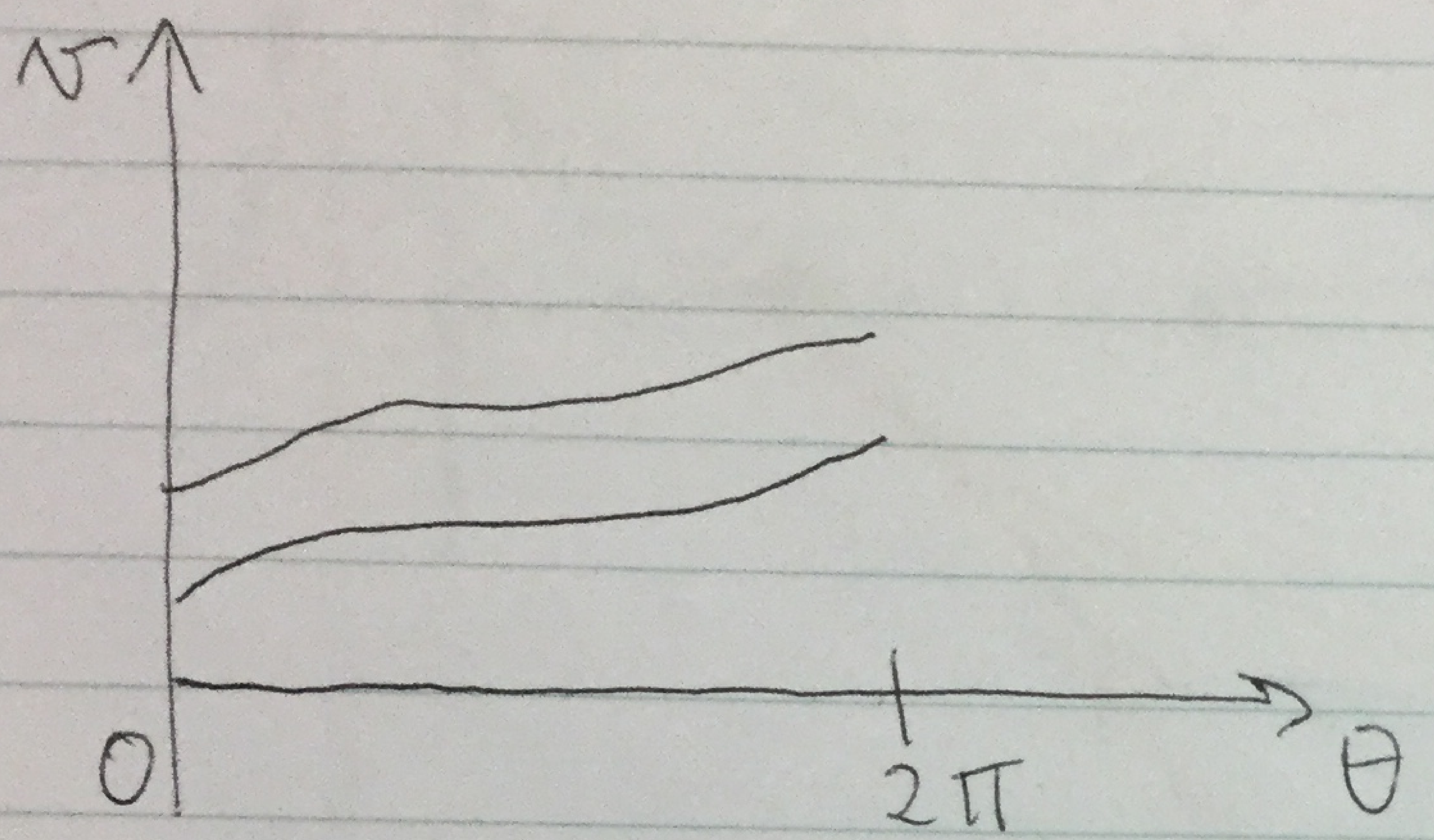
Decrease k a little and the picture needs to change continuously, thus the periodic solution green can not dissapear



5) Prove there is only one periodic solution (5)

$k > 1$

$E(\theta, v) = \frac{v^2}{2} - \cos\theta$



(θ, v) solution with $v(0) > 0$. think of $v = v(\theta) = v(\theta(t))$. then

$$\frac{dE}{dt} = \frac{\partial E}{\partial \theta}\theta' + \frac{\partial E}{\partial v} v' = \sin\theta\, v + v(-\sin\theta - bv + k)$$

$$= v(-bv + k)$$

$$\frac{dE}{d\theta} = \frac{\frac{dE}{dt}}{\frac{d\theta}{dt}} = \frac{v(-bv+k)}{v} = -bv + k$$

If (θ_1, v_1) & (θ_2, v_2) are two solutions with

$\theta_1(0) = \theta_2(0) = 0$ & $v_1(0) < v_2(0)$ then $v_1(t) < v_2(t)$ for all t also θ

$$\int_0^{2\pi} \frac{dE_1}{d\theta} = -b\int_0^{2\pi} v_1(\theta)\, d\theta + k2\pi >$$

$$> -b\int_0^{2\pi} v_2(\theta)\, d\theta + 2\pi k = \int_0^{2\pi} \frac{dE_2}{d\theta}, \text{ then}$$

the change of energy can not be zero for both

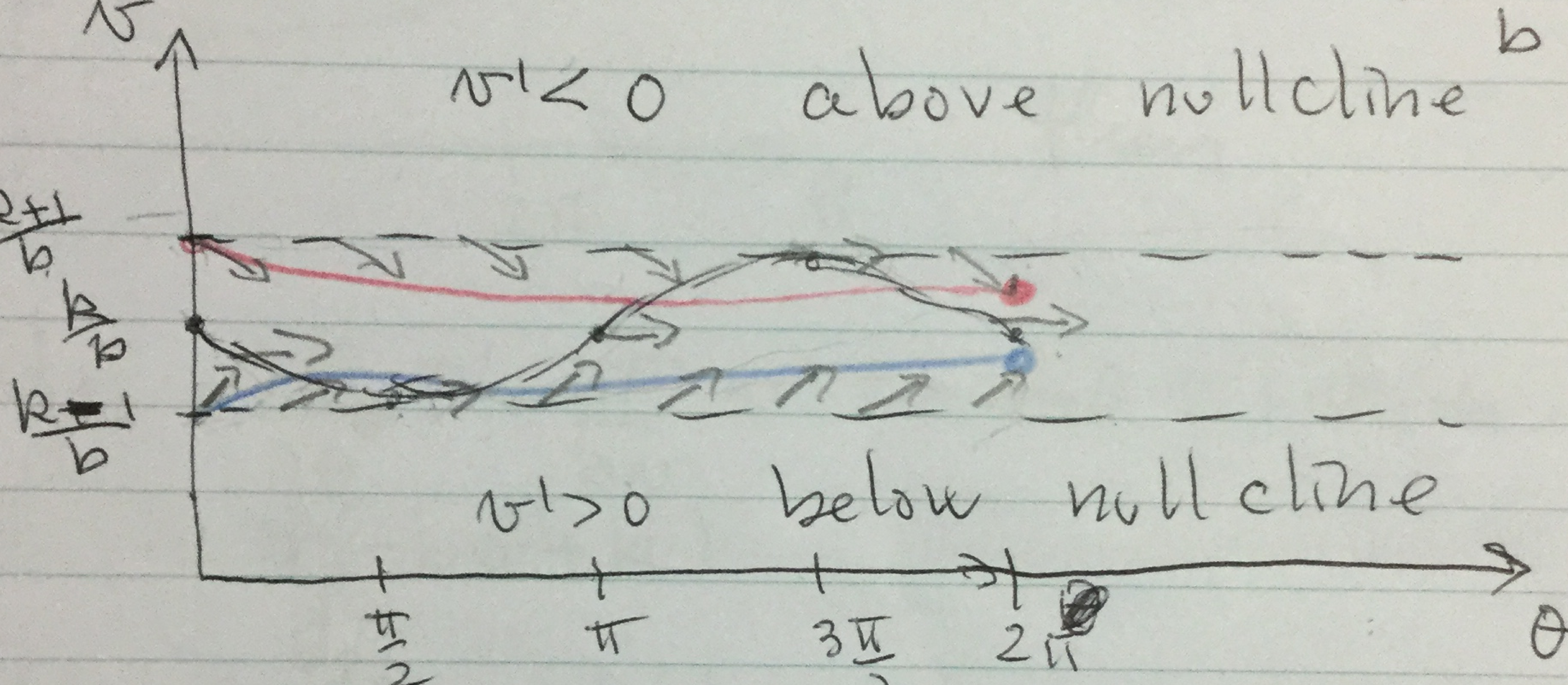
$k>1$ no fixed point

$k=1$ one fixed point

$0\leq k<1$ two fixed points.

3) $k>1$ • v-nullcline $v=\frac{k-\sin\theta}{b}$

$v'<0$ above nullcline

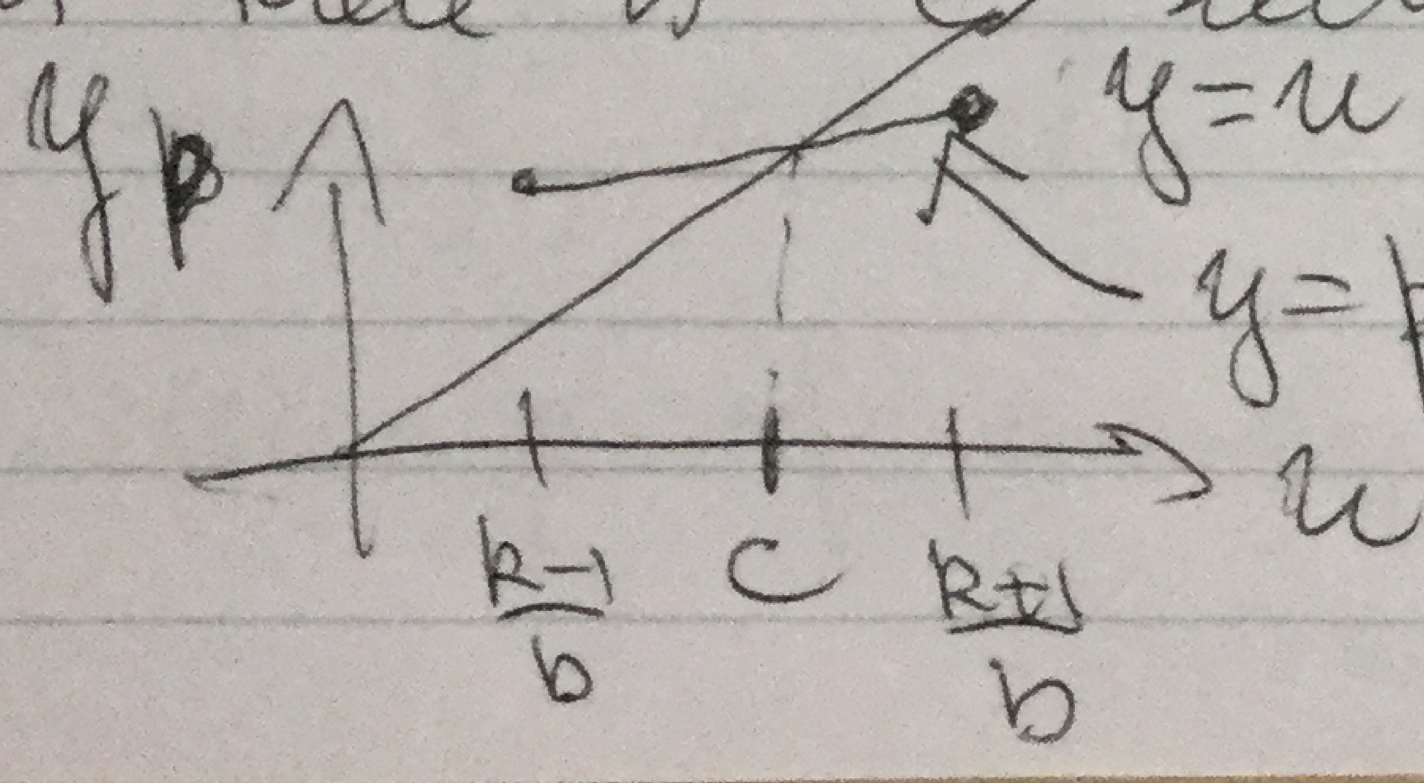


$v'>0$ below nullcline

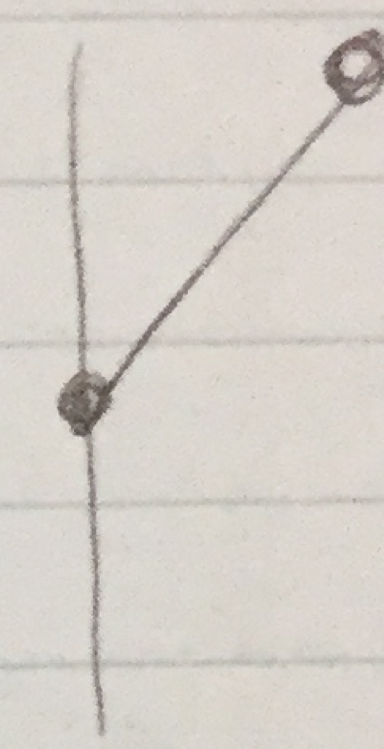
• $p(u)=v(T)$ where (θ,v) is a solution of the system, $\theta(0)=0$ $v(0)=u$ & $\theta(T)=2\pi$

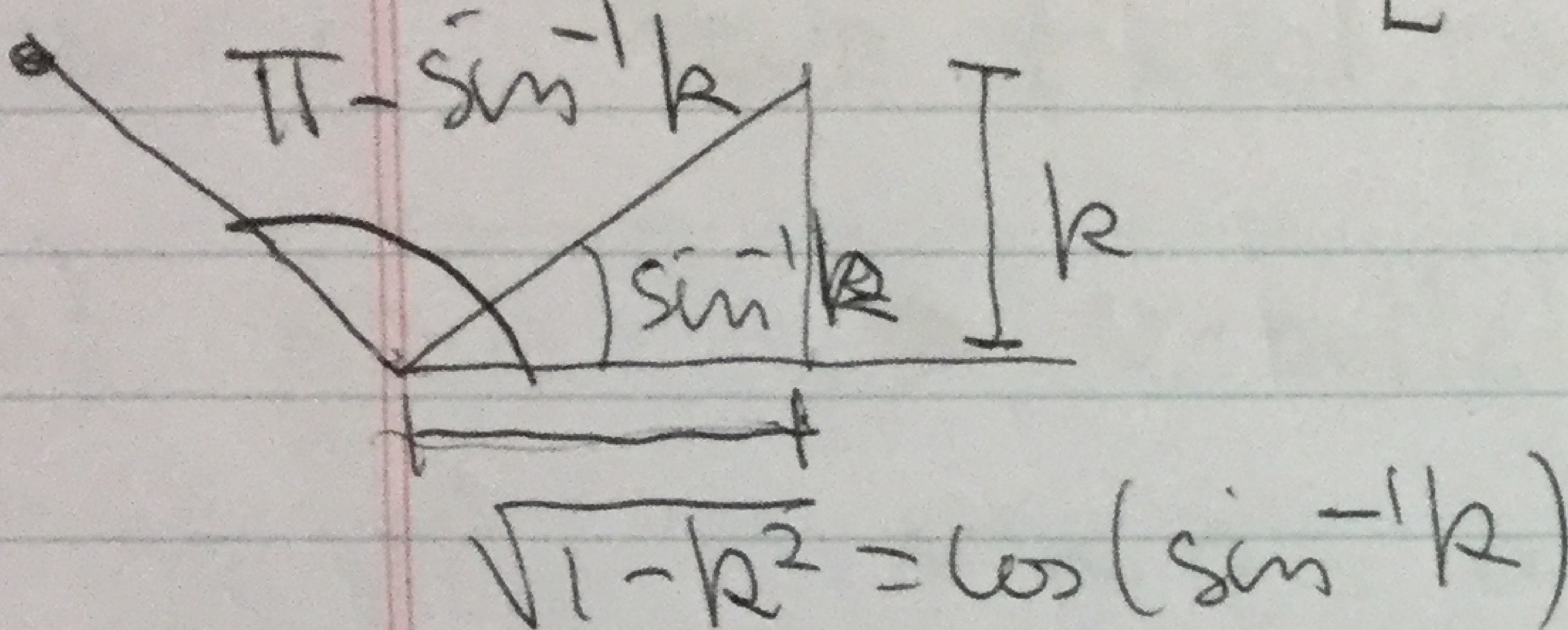
p is the Poincare map. The above graph implies $p\left(\frac{k-1}{b}\right)\geq\frac{k-1}{b}$ & $p\left(\frac{k+1}{b}\right)\leq\frac{k+1}{b}$

then there is c such that $p(c)=c$



$$A = (\pi - \sin^{-1}k; 0) = \begin{bmatrix} 0 & 1 \\ \sqrt{1-k^2} & -b \end{bmatrix}$$





$$P(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ \sqrt{1-k^2} & -b-\lambda \end{bmatrix} =$$

$$= \lambda^2 + b\lambda - \sqrt{1-k^2} = 0$$

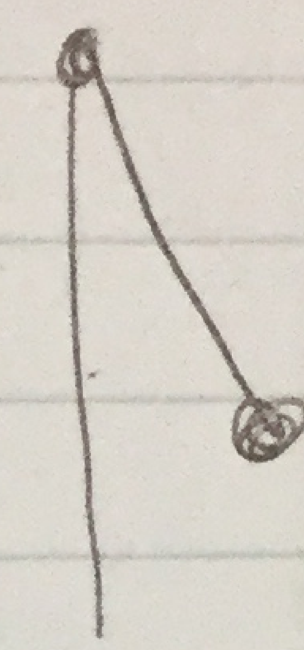
$$\lambda = \frac{-b \pm \sqrt{b^2 + 4\sqrt{1-k^2}}}{2}$$ one positive

eigenvalue and one negative (unless $k=1$, one negative and one equal to 0)

$(\pi - \sin^{-1}k, 0)$ is a saddle

2) Determine the regions in the bk-parameter plane for which the are different numbers of equilibrium points

$\boxed{\text{at } (\theta, v) = (\sin^{-1}k, 0)}$



$$A(\sin^{-1}k, 0) = \begin{bmatrix} 0 & 1 \\ -\sqrt{1-k^2} & -b \end{bmatrix}$$

Eigenvalues of A

$$P(\lambda) = \det[A - \lambda I] = \begin{bmatrix} -\lambda & 1 \\ -\sqrt{1-k^2} & -b-\lambda \end{bmatrix} =$$

$$= \lambda^2 + b\lambda + \sqrt{1-k^2} = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4\sqrt{1-k^2}}}{2}$$

Case $b^2 \geq 4\sqrt{1-k^2} > 0$ then both eigenvalues are real and negative (($\sin^{-1}k, 0$) is a ~~source~~ sink)

Case $b^2 < 4\sqrt{1-k^2}$ λ is complex with real part negative ($\sin^{-1}k, 0$) is a stable spiral.

at $(\theta, v) = (\pi - \sin^{-1}k, 0)$