Reminder Hamiltonian Systems

$$x' = \frac{\partial H}{\partial y} \qquad y' = -\frac{\partial H}{\partial x}$$

where H is the Hamiltonian

Key: H = constant along trajectories.

this means that if we know the level curves of H, we know the trajectories of the system.

Example $\quad x' = y \qquad y' = -x^3 + x$

$$H = \frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}$$

Critical points of H were $\begin{bmatrix} -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$DF = \begin{bmatrix} \frac{\partial^2 H}{\partial x \partial y} & \frac{\partial^2 H}{\partial y^2} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial x \partial y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{bmatrix}$$

This is to find the type of fixed

points

Fixed Point	$\begin{bmatrix} -1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
DF	$\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$
eigenvalues	$\pm i\sqrt{2}$	± 1	$\pm i\sqrt{2}$
type	center	saddle	center

Let's draw the level sets of H

$$H = \frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}$$

$$A = \begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial y} \\ \frac{\partial^2 H}{\partial x \partial y} & \frac{\partial^2 H}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 3x^2-1 & 0 \\ 0 & 1 \end{bmatrix}$$

Critical point	$\begin{bmatrix} -1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
A	$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$
eigenvalues	$\lambda = 1, 2$	$\lambda = -1, 1$	$\lambda = 1, 2$

min saddle min

$H\left(\begin{bmatrix}0\\0\end{bmatrix}\right)=0$

For ~~x, y~~ $\begin{bmatrix}x\\y\end{bmatrix}$ near $\begin{bmatrix}0\\0\end{bmatrix}$

~~$H(x,y) \approx$~~

$$H\left(\begin{bmatrix}x\\y\end{bmatrix}\right) \approx \underbrace{H\left(\begin{bmatrix}0\\0\end{bmatrix}\right)}_{=0} + \underbrace{\nabla H\left(\begin{bmatrix}0\\0\end{bmatrix}\right)}_{=0} \cdot \begin{bmatrix}x\\y\end{bmatrix} +$$

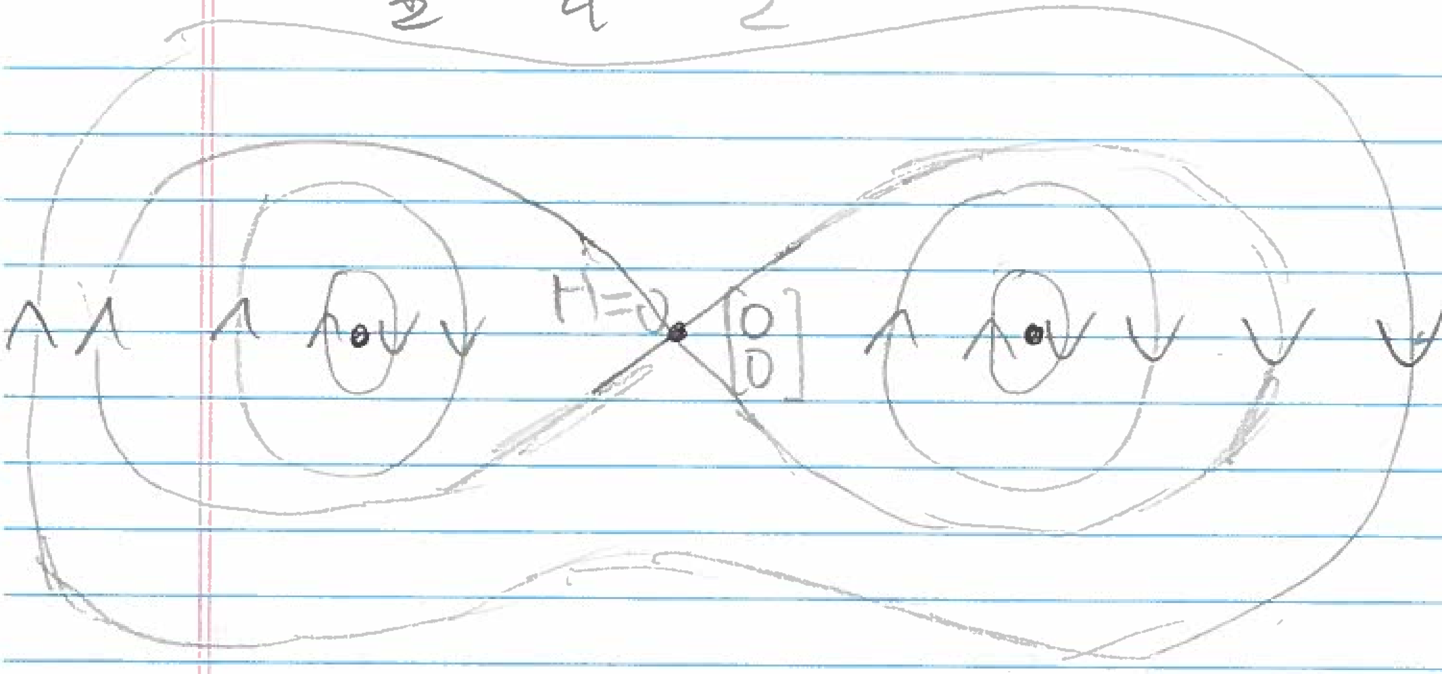
$$+\frac{1}{2}\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}\frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial y}\\ \frac{\partial^2 H}{\partial x \partial y} & \frac{\partial^2 H}{\partial^2 y}\end{bmatrix}_{\text{at } \begin{bmatrix}0\\0\end{bmatrix}}\begin{bmatrix}x\\y\end{bmatrix} =$$

$$=\frac{1}{2}\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{2}(-x^2+y^2)$$

$H \approx \frac{1}{2}(-x^2+y^2)$ for $\begin{bmatrix}x\\y\end{bmatrix}$ near $\begin{bmatrix}0\\0\end{bmatrix}$

$\frac{1}{2}(-x^2+y^2)=\frac{1}{2}(y-x)(y+x)$

$H = \frac{y^2}{2} + \frac{x^4}{4} - \frac{x^2}{2}$



<u>Obs</u>: $x' = \frac{\partial H}{\partial y}$ $\quad y' = -\frac{\partial H}{\partial x}$

$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is a fixed point

$$DF = \begin{bmatrix} \frac{\partial^2 H}{\partial x \partial y} & \frac{\partial^2 H}{\partial y^2} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial y \partial x} \end{bmatrix}$$

Characteristic polynomial

$$P(\lambda) = \det \begin{bmatrix} \frac{\partial^2 H}{\partial x \partial y} - \lambda & \frac{\partial^2 H}{\partial y^2} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial y \partial x} - \lambda \end{bmatrix} =$$

$$= \left(\frac{\partial^2 H}{\partial x \partial y} - \lambda\right)\left(-\frac{\partial^2 H}{\partial y \partial x} - \lambda\right) + \frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2} =$$

$$= \lambda^2 - \left[\left(\frac{\partial^2 H}{\partial x \partial y}\right)^2 - \frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2}\right]$$

evaluated at $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

roots, If $\left(\frac{\partial^2 H}{\partial x \partial y}\right)^2 - \frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2} > 0$

then $\lambda = \pm \sqrt{\left(\frac{\partial^2 H}{\partial x \partial y}\right)^2 - \frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2}}$

evaluated at $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

If $\left(\frac{\partial^2 H}{\partial x \partial y}\right)^2 - \frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2} < 0$ at $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

then $\lambda = \pm i \sqrt{\frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2} - \frac{\partial^2 H}{\partial x \partial y}}$

Example

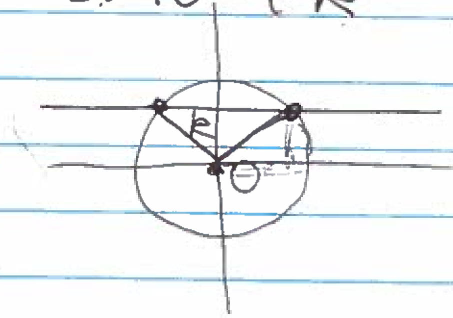
$\theta' = v$

$v' = -\sin\theta - bv + k$

~~1) Fin~~ $k \geq 0 \quad b \geq 0$

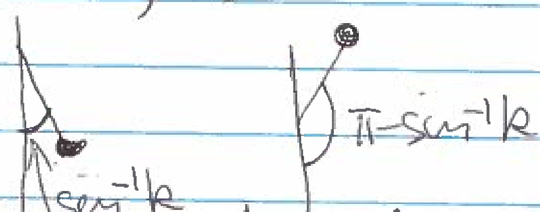
1) Find all equilibrium points

$v = 0 \qquad 0 = -\sin\theta + k$



If $0 \leq k \leq 1$, then $\theta = \sin^{-1}k + 2\pi n$ or $\pi - \sin^{-1}k + 2\pi n$

n integer



If $k > 1$ no fixed points

Stability $DF = \begin{bmatrix} 0 & 1 \\ -\cos\theta & -b \end{bmatrix}$

at $\theta = \sin^{-1}k$ $\quad DF = \begin{bmatrix} 0 & 1 \\ -\sqrt{1-k^2} & -b \end{bmatrix}$

$\cos\theta = \sqrt{1-k^2}$

Eigenvalues

$$P(\lambda) = \det \begin{bmatrix} -\lambda & 1 \\ -\sqrt{1-k^2} & -b-\lambda \end{bmatrix} = \lambda^2 + b\lambda + \sqrt{1-k^2} = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4\sqrt{1-k^2}}}{2}$$

$$b^2 - 4\sqrt{1-k^2} < b^2$$

then $\sqrt{b^2 - 4\sqrt{1-k^2}} < b$

Case $b^2 - 4\sqrt{1-k^2} > 0$

$$\sqrt{b^2 - 4\sqrt{1-k^2}} < b \Rightarrow \frac{-b + \sqrt{b^2 - 4\sqrt{1-k^2}}}{2} < 0$$

$$\frac{-b - \sqrt{b^2 - 4\sqrt{1-k^2}}}{2} < 0$$

Case $b^2 - 4\sqrt{1-k^2} < 0$