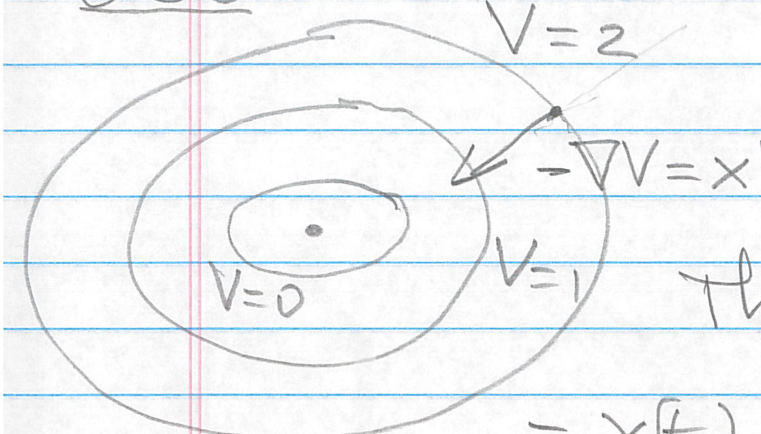
Gradient Systems

$$x' = -\nabla V$$

Obs level sets of V

$V = c$



The trajectories $x = x(t)$ are perpendicular to the level sets of V, and they point in the direction that V decreases.

Theorem $x' = -\nabla V$

1) The trajectories are perpendicular to the level sets of V

2) the critical points of V are

the equilibrium points of the system

3) If a critical point is an isolated minimum of V, then this point is an asymptotically stable fixed point of the system.

Example: $V = x^2(x-1)^2 + y^2$

$x' = -\nabla V$

$$\boxed{x' = -2x(x-1)^2 - 2x^2(x-1)} = \overline{\underline{-2x(x-1)(2x-1)}}$$

$$\boxed{y' = -2y}$$

Equilibrium points

$0 = -2x(x-1)(2x-1)$ $\quad x = 0, 1, \frac{1}{2}$

$0 = -2y$ $\quad y = 0$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$DF = \begin{bmatrix} -\frac{\partial^2 V}{\partial x^2} & -\frac{\partial^2 V}{\partial x \partial y} \\ -\frac{\partial^2 V}{\partial x \partial y} & -\frac{\partial^2 V}{\partial y^2} \end{bmatrix}$$ (3)

x	0	$1/2$	1
$-\frac{\partial^2 V}{\partial x^2}$	-2	1	-2

because

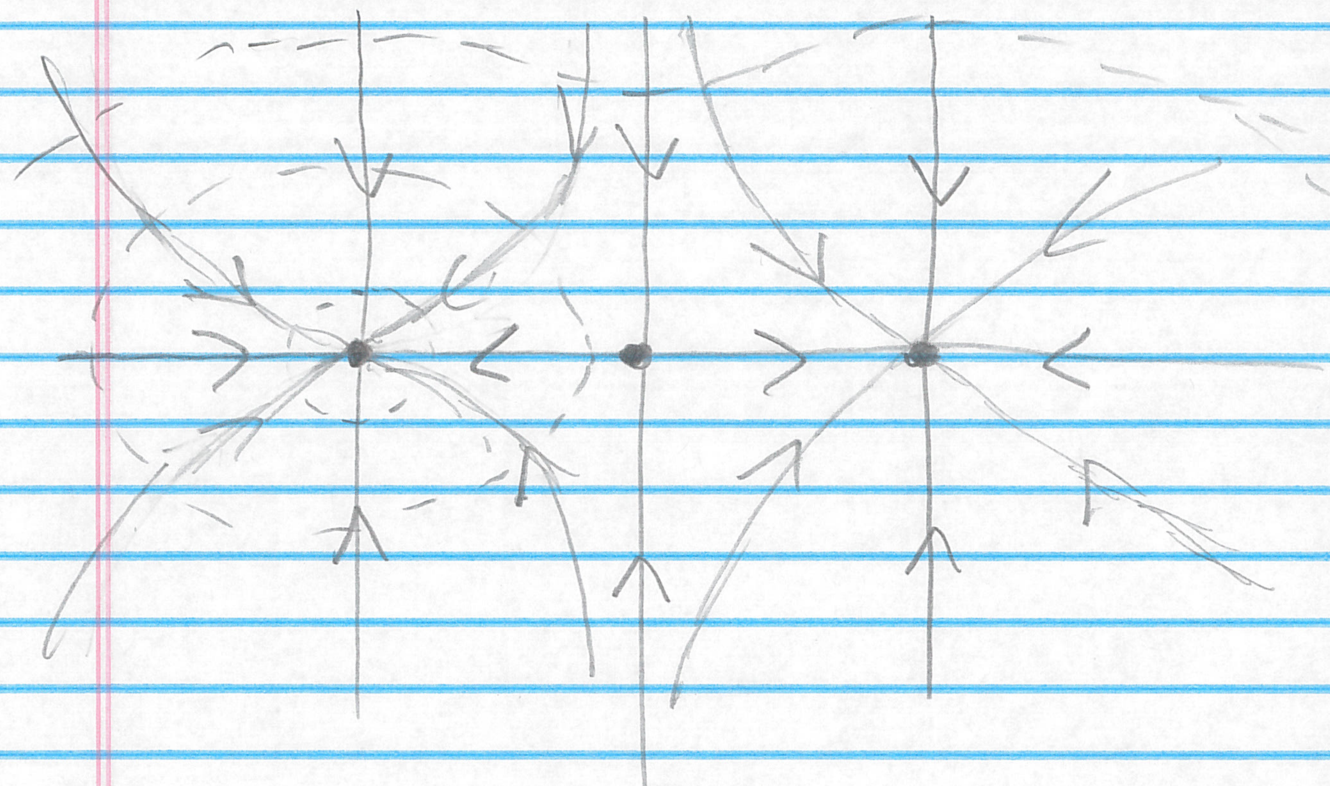
$$-\frac{\partial^2 V}{\partial x^2} = -2\left[(x-1)(2x-1) + 2x(x-1) + x(2x-1)\right]$$

$$-\frac{\partial^2 V}{\partial x \partial y} = 0 \qquad -\frac{\partial^2 V}{\partial y^2} = -2 \qquad DF = \begin{bmatrix} -\frac{\partial^2 V}{\partial x^2} & 0 \\ 0 & -\frac{\partial^2 V}{\partial y^2} \end{bmatrix}$$

Fixed points	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
Eigenvalues & Eigenvectors	-2 $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	-2 $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	-2 $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
	-2 $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	1 $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	-2 $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

With dotted lines (4)

Phase plane the level sets of V



Obs: If $A \in \mathbb{R}^{n \times n}$ and A is symmetric, then all its eigenvalues are real

Obs: $x' = -\nabla V$. The the eigenvalues of DF at any fixed point are real where, $F = -\nabla V$. Because $DF = -HV$

In two dimensions,

$$-HV = \begin{bmatrix} -\frac{\partial^2 V}{\partial x^2} & -\frac{\partial^2 V}{\partial x \partial y} \\ -\frac{\partial^2 V}{\partial x \partial y} & -\frac{\partial^2 V}{\partial y^2} \end{bmatrix}$$

which is a symmetric matrix. thus the fixed points of $x' = -\nabla V$ are never spirals or centers.

Hamiltonian Systems

These are systems of the form

$$x' = \frac{\partial H}{\partial y}$$

$$y' = -\frac{\partial H}{\partial x}$$

where $H: \mathbb{R}^2 \longrightarrow \mathbb{R}$. H is called the Hamiltonian

Example 1)

$$x' = y$$

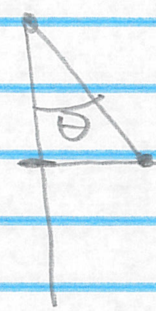
$$y' = -kx$$

$$H(x,y) = \frac{1}{2}y^2 + \frac{k}{2}x^2$$

$$\frac{\partial H}{\partial y} = y \qquad -\frac{\partial H}{\partial x} = -kx \quad \checkmark$$

H is a Hamiltonian for this system.

2)



$$\theta' = v$$

$$v' = -\sin\theta$$

$$H(\theta, v) = \frac{1}{2}v^2 + 1 - \cos\theta$$

$$\frac{\partial H}{\partial v} = v \qquad -\frac{\partial H}{\partial \theta} = -\sin\theta \quad \checkmark$$

H is a Hamiltonian for this system

<u>Prop</u>: $x' = \frac{\partial H}{\partial y}$ then H is constant along
$y' = -\frac{\partial H}{\partial x}$ the trajectories

<u>proof</u>: Let $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ be a solution of this system

$$\frac{d}{dt} H(x(t), y(t)) = \frac{\partial H}{\partial x} x' + \frac{\partial H}{\partial y} y' =$$

$$= \frac{\partial H}{\partial x}\left(\frac{\partial H}{\partial y}\right) + \frac{\partial H}{\partial y}\left(-\frac{\partial H}{\partial x}\right) = 0$$

<u>Example</u> $x' = y$

$y' = -x^3 + x$

Is there a Hamiltonian?

$$x' = y = \frac{\partial H}{\partial y} \longrightarrow H = \frac{1}{2}y^2 + g(x)$$

$$y' = -x^3 + x = -\frac{\partial H}{\partial x} = -\frac{dg(x)}{dx} \rightarrow g = \frac{x^4}{4} - \frac{x^2}{2}$$

$$H = \frac{1}{2}y^2 + \frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{4}$$

<u>Fixed points</u> $\quad y = 0$

$$-x^3 + x = 0$$

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad x' = \frac{\partial H}{\partial y}$$

$$y' = -\frac{\partial H}{\partial x}$$

$$DF = \begin{bmatrix} \frac{\partial^2 H}{\partial x \partial y} & \frac{\partial^2 H}{\partial y^2} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial x \partial y} \end{bmatrix}$$

$$DF = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad DF\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \qquad \lambda = \pm i\sqrt{2}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad DF\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \lambda = \pm 1$$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad DF\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \quad \lambda = \pm i\sqrt{2}$

Eigen vectors of $DF\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$-1 \quad DF - (-1)I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$1 \quad DF - I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$