1

Pendulum

$$\theta'' + \sin\theta = -\nu\dot{\theta}$$

Change to first order system

$$\theta' = v$$

$$v' = -\sin\theta - \nu v$$

$$L(\theta, v) = \frac{1}{2}v^2 - \cos\theta$$

Is L a Lyapunov function for $\begin{bmatrix}\theta \\ v\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$

We need

$\nabla L \cdot F < 0$ for all θ, v near the fixed point

$$\nabla L = \begin{bmatrix}\frac{\partial L}{\partial \theta} \\ \frac{\partial L}{\partial v}\end{bmatrix} \qquad F = \begin{bmatrix} v \\ -\sin\theta - \nu v\end{bmatrix}$$

$$\frac{\partial L}{\partial \theta} = \sin\theta \qquad \frac{\partial L}{\partial v} = v$$

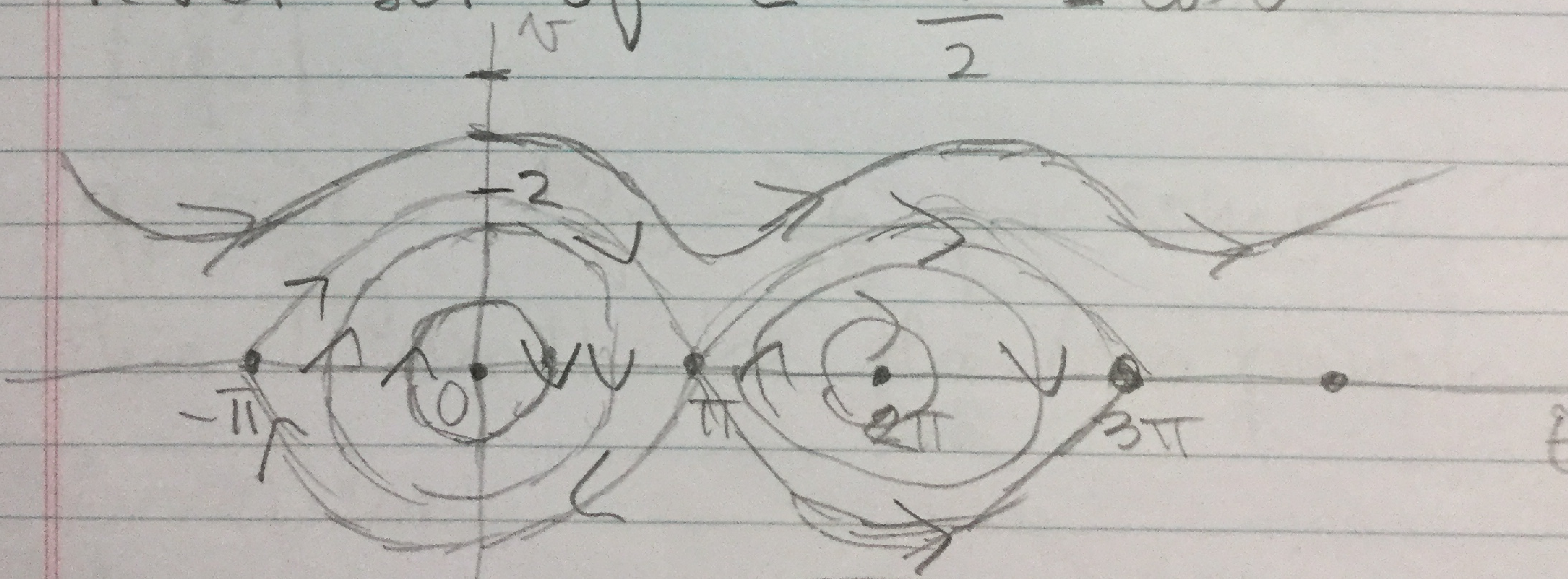
$$\nabla L \cdot F = \begin{bmatrix} \sin\theta \\ v \end{bmatrix} \cdot \begin{bmatrix} v \\ -\sin\theta - \gamma v \end{bmatrix} =$$

$$= -\gamma v^2 < 0 \quad \text{if } v \neq 0$$

$\boxed{\text{Case } \gamma = 0}$. If $\begin{bmatrix} \theta(t) \\ v(t) \end{bmatrix}$ is a solution

then $\frac{d L(\theta(t), v(t))}{dt} = -\gamma\, v^2(t) = 0$

Each trajectory is restricted to a level set of $L = \frac{v^2}{2} - \cos\theta$



$$\frac{v^2}{2} - \cos\theta = k \qquad v = \pm\sqrt{2(k + \cos\theta)}$$

$f(x,y)$ a level set of f is
$\{(x,y): f(x,y) = k\}$ k is a
constant

Example $\quad x' = -x^3$

near $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad y' = -y(x^2+z^2+1)$

$z' = -\sin z$

Can we find g_1, g_2, g_3 functions of (x,y,z) such that

$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} \cdot F < 0$ also $g_1 = \frac{\partial L}{\partial x}$

$g_2 = \frac{\partial L}{\partial y}$ $\quad g_3 = \frac{\partial L}{\partial z}$ for some L?

This L will be the Lyapunov function

$$L = \frac{x^2 + y^2 + z^2}{2}$$

$$\frac{dL}{dt} = x x' + y y' + z z' =$$

$$= x(-x^3) + y(-y(x^2+z^2+1)) +$$

$$+ z(-\sin z) = -[x^4 + y^2(x^2+z^2+1)$$

$$+ z \sin z] < 0 \quad \text{if} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and $|z| < \pi$ also

$$L(x,y,z) > 0 \text{ if } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \ \& \ |z| < \pi$$

$$L(0,0,0) = 0$$

Thus, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a stable (asympto-

tically) fixed point.

Gradient systems

These are systems of the form

$$x' = -\nabla V$$

for some $V = V(x)$

Ex: $x' = -2x(x-1)(2x-1)$

$$y' = -2y$$

Is there a V such that

$$\begin{cases} -2x(x-1)(2x-1) = -\dfrac{\partial V}{\partial x} \text{ and} \\ -2y = -\dfrac{\partial V}{\partial y} \end{cases}$$

Yes $\boxed{V = x^4 - 2x^3 + x^2 + y^2}$

$2x(x-1)(2x-1) = 2x(2x^2 - 3x + 1) =$

$= 4x^3 - 6x^2 + 2x$

<u>Obs</u>: $x' = -\nabla V$.

1) Then the fixed points of this system are the critical points of V, i.e. x^* such that $\nabla V(x^*) = 0$.

2) If x^* is a local minimum or a local maximum of V, then x^* is a fixed point of $x' = -\nabla V$

3) If $x(t)$ is a solution of $x' = -\nabla V(x)$, then

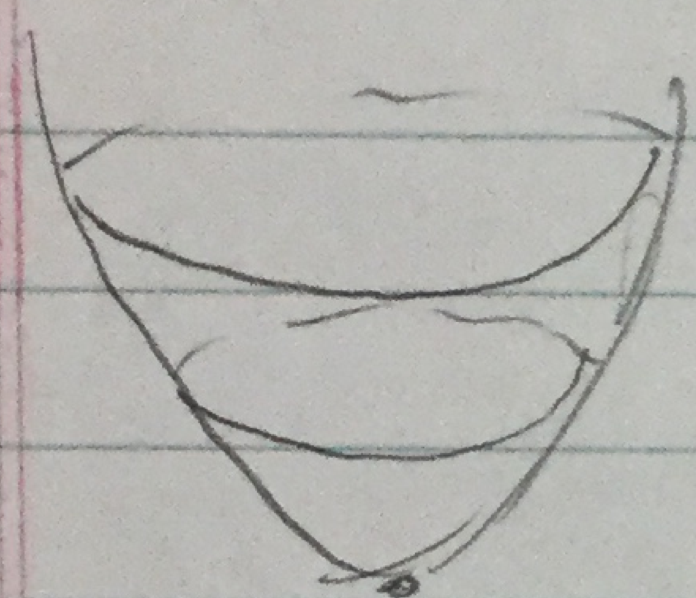
$$\frac{d}{dt} V(x(t)) = \nabla V(x(t)) \cdot x'(t) =$$

$$= \nabla V(x(t)) \cdot \left(-\nabla V(x(t))\right) =$$

$$= -\|\nabla V(x(t))\|^2 \leq 0$$

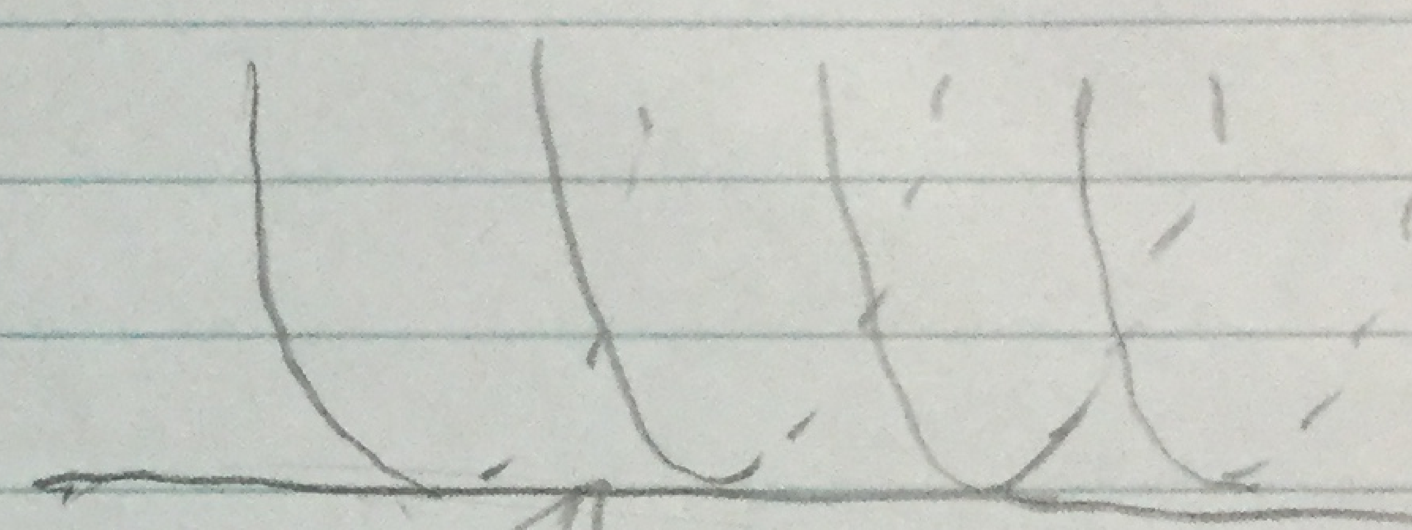
If x^* is a "strict" local minimum of V, then

$\|\nabla V(x)\| \neq 0$ for all x near to x^* but $x \neq x^*$



x^*

strict local minimum



x^*

<u>Lesson</u>: If we have a gradient system $x' = -\nabla V(x)$, the asymptotically stable fixed points of this system are the "strict" local minima of $V(x)$.