Liapunov functions

Idea: If we know that the energy of a system "always" decreases, then the system goes to a state of a local minimum of the energy

Def: Let $x' = F(x)$ be a system of differential equations. Let x^* be a fixed point, i.e. $F(x^*) = 0$

Let U be an open set that contains x^*. Let $L : U \longrightarrow \mathbb{R}$. L is a Liapunov function for x^* if:

1) $L(x^*) < L(x)$ for all $x \in U - \{x^*\}$

2) $\nabla L(x) \cdot F(x) < 0$ for all $x \in U - \{x^*\}$

Note: Assume $x = x(t)$ is a solution of $x' = F(x)$. Then

$$\boxed{\frac{d}{dt} L(x(t)) = \nabla L(x(t)) \cdot x'(t) = \nabla L(x(t)) \cdot F(x(t))}$$

Recall: $\nabla L(x) = \begin{bmatrix} \frac{\partial L}{\partial x_1} \\ \frac{\partial L}{\partial x_2} \\ \vdots \\ \frac{\partial L}{\partial x_n} \end{bmatrix}$

$$v \cdot w = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

Obs: From the note above,

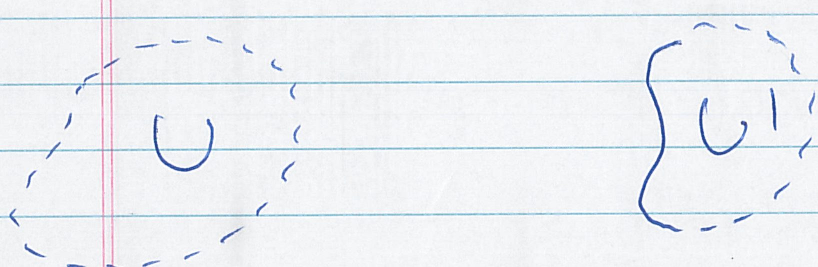
$$\frac{d}{dt} L(x(t)) = \nabla L(x(t)) \cdot F(x(t)) < 0$$

for all t. This means that L

decreases along trajectories.

Note that we need $x(t)$ to be in U for all t.

<u>Recall</u>: $U \subset \mathbb{R}^n$ is an open set if U does not contain any point in its "border" or "boundary".



U is open U' is not open

<u>Theorem</u>: Let x^* be a fixed point of $x' = F(x)$. If there is a Liapunov function for x^*, then x^* is "asymptotically" stable, i.e. if $x(0)$ is close enough to x^*, then $x(t) \longrightarrow x^*$

<u>Example</u> $x' = (\varepsilon x + 2y)(z+1)$

$$y' = (-x + \varepsilon y)(z+1)$$

$$z' = -z^3$$

the origin is a fixed point.

$$DF = \begin{bmatrix} \varepsilon(z+1) & 2(z+1) & (\varepsilon x + 2y) \\ -(z+1) & \varepsilon(z+1) & (-x+\varepsilon y) \\ 0 & 0 & -2z^2 \end{bmatrix}$$

$$DF\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \varepsilon & 2 & 0 \\ -1 & \varepsilon & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P(\lambda) = (-\lambda)\left((\varepsilon - \lambda)^2 + 2\right) = 0 \qquad \begin{matrix} \lambda = 0 \text{ or} \\ \lambda = \varepsilon \pm i\sqrt{2} \end{matrix}$$

~~$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is stable if $\varepsilon < 0$, it is unstable if $\varepsilon > 0$. We do not know if $\varepsilon = 0$. We do not~~

$\begin{bmatrix}0\\0\\0\end{bmatrix}$ is unstable if $\varepsilon > 0$, but we do not know its stability if $\varepsilon \le 0$.

We try to find a Liapunov function of the form $L = ax^2 + by^2 + cz^2$

$\nabla L \cdot F$

$$\begin{bmatrix}2ax\\2by\\2cz\end{bmatrix} \cdot \begin{bmatrix}(\varepsilon x + 2y)(z+1)\\(-x+\varepsilon y)(z+1)\\-z^3\end{bmatrix} = 2ax(\varepsilon x + 2y)$$

$$(z+1) + 2by(-x+\varepsilon y)(z+1) + 2cz(-z^3) =$$

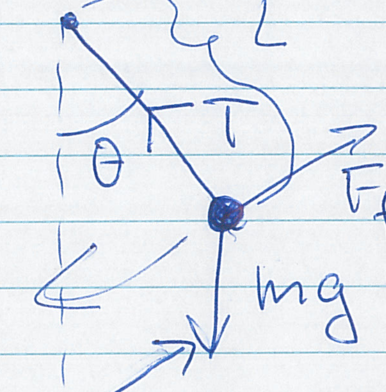
$b = 2 \quad a = 1 \quad c = 1$

$$= 2\varepsilon x^2(z+1) + 4\varepsilon y^2(z+1) - 2z^4$$

$$\frac{d}{dt} L(x(t), y(t), z(t)) = 2\varepsilon(x^2 + 2y^2)(z+1) - 2z^4$$

$z > -1$ and $\varepsilon < 0$ then $\nabla L \cdot F < 0$ for all $(x, y, z) \neq (0,0,0)$. then $(0,0,0)$ is asymptotically stable

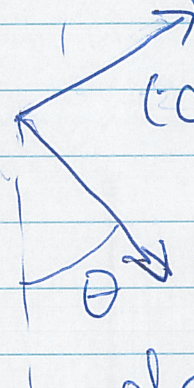
<u>Example</u> $\vec{T} = -e_\theta T \qquad T > 0$



$F_g = -e_2 mg \qquad e_2 = (0,1)$

$x = L e_\theta$

$F_f = -\nu e_\theta^\perp L \theta'$



$(\cos\theta, \sin\theta) = e_\theta^\perp$

$e_\theta = (\sin\theta, -\cos\theta)$

$$\frac{de_\theta}{d\theta} = (\cos\theta, \sin\theta) \qquad \frac{de_\theta^\perp}{d\theta} = -e_\theta$$

$$x' = \frac{dx}{dt} = L \frac{d}{dt} e_\theta = L \frac{d}{d\theta} e_\theta \, \theta' = L e_\theta^\perp \theta'$$

$-\nu e_\theta^\perp L \theta'$

$$x'' = L(-e_\theta)(\theta')^2 + L e_\theta^\perp \theta''$$

e_2

Newton: $mL\left(e_\theta^\perp \theta'' - e_\theta (\theta')^2\right) = -e_\theta T - mg$

dot product with $e_\theta^\perp$

$$\boxed{mL\theta'' = -mg\sin\theta - \nu L\theta'}$$

Take $g = L = 1$

$$\theta' = v$$

$$v' = -\frac{g}{L}\sin\theta - \frac{\gamma}{L} v$$

~~Fixed points~~

$\cancel{v = 0}$ $\qquad \theta' = v$

$$v' = -\sin\theta - \gamma v$$