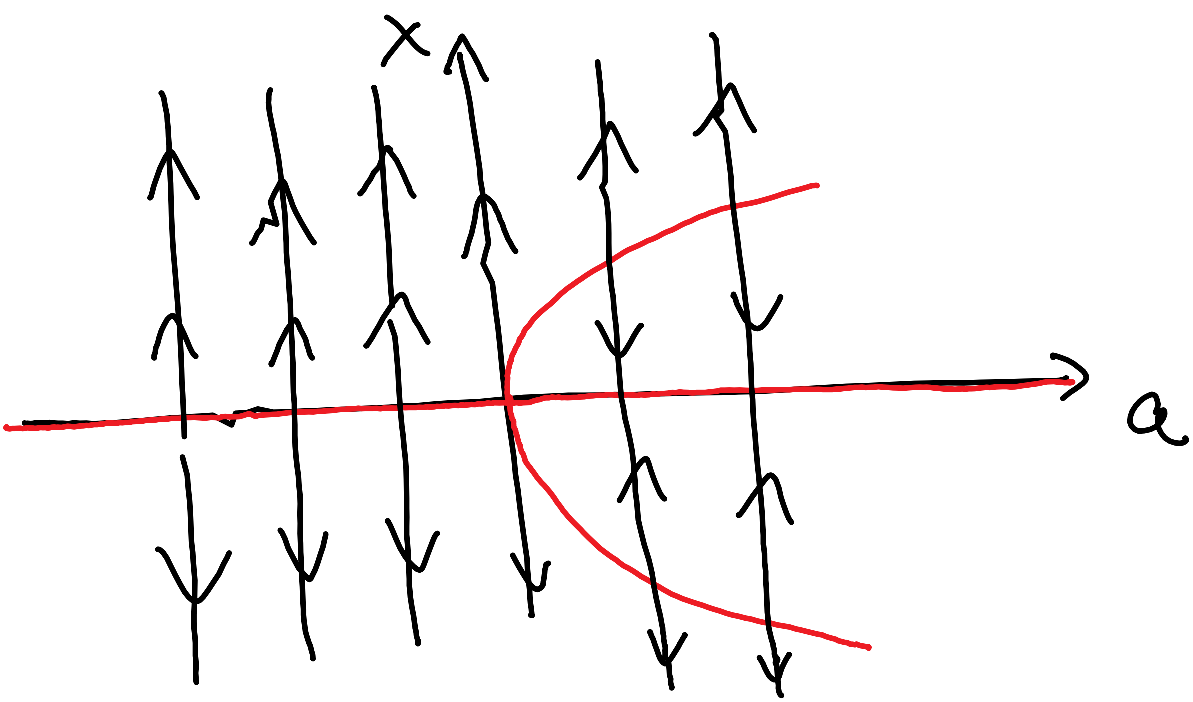
Pitchfork bifurcation

$$x' = x^3 - ax$$

$x^3 - ax = 0$ gives all the fixed points

$x(x^2 - a) = 0$

$x = 0 \qquad x = \pm\sqrt{a}$ if $a > 0$



Stability of fixed points of non-linear systems

$$x_1' = f_1(x_1, \ldots, x_n)$$
$$\vdots \qquad \vdots$$
$$x_n' = f_n(x_1, \ldots, x_n)$$

$$X' = F(x)$$

Fixed point are the solutions of $F(x)=0$

Let x^* be a fixed point, $F(x^*)=0$

Let $x = x^* + \Delta x$

$$F(x) = F(x^* + \Delta x) \cong \underbrace{F(x^*)}_{=0} + DF(x^*)\Delta x$$

$$DF = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \partial f_n & \partial f_n & & \partial f_n \end{bmatrix} = A$$

$$\left[\frac{\partial f_n}{\partial x_1} \; \frac{\partial f_n}{\partial x_2} \; \cdots \; \frac{\partial f_n}{\partial x_n} \right]$$

$$x' = (\Delta x)' = F(x) \approx A(\Delta x)$$

$$(\Delta x)' = A(\Delta x)$$

<u>Theorem</u>: Let x^* be a fixed point of $x' = F(x)$. Let $A = DF(x^*)$.

1) If all the eigenvalues of A have negative real part, then x^* is stable

2) If one of the eigenvalues of A has positive real part, then x^* is unstable.

3) Otherwise, we do not know.

<u>Examples</u>: Find the fixed points and their stability.

1) $x' = 1-2xy$

$y' = 2xy - y$

$1-2xy=0$

$2xy - y = 0 \longrightarrow y(2x-1)=0$

$y=0$

$1=0$ No!

$x = \frac{1}{2}$

$1-y=0 \quad y=1$

Yes!

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ is the only fixed point.

$$DF = \begin{bmatrix} -2y & -2x \\ 2y & 2x-1 \end{bmatrix}$$

$$DF\left(\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 & -1 \\ 2 & 0 \end{bmatrix}$$

Eigenvalues

$P(\lambda) = (-2-\lambda)(-\lambda) + 2 = \lambda^2 + 2\lambda + 2 =$

$= (\lambda+1)^2 + 1$

$\lambda = -1 \pm i$ $\quad$ $Re(\lambda) < 0$ for all

the eigenvalues

$\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ is stable (stable spiral)

2) $x' = -3x + y^2 + 2$

$y' = x^2 - y^2$

$-3x + y^2 + 2 = 0$

$x^2 - y^2 = 0$ $\quad$ $x^2 = y^2$ plug into

first eq $\quad$ $x^2 - 3x + 2 = 0$

$x = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = \begin{cases} 2 \\ 1 \end{cases}$

Fixed points $\begin{bmatrix} 1 \\ \end{bmatrix}$ $\begin{bmatrix} 1 \\ \end{bmatrix}$ $\begin{bmatrix} 2 \\ \end{bmatrix}$ $\begin{bmatrix} 2 \\ \end{bmatrix}$

Fixed points $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$DF = \begin{bmatrix} -3 & 2y \\ 2x & -2y \end{bmatrix}$$

$\boxed{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$ $DF\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -3 & -2 \\ 2 & 2 \end{bmatrix}$

$$P(\lambda) = (-3-\lambda)(2-\lambda) + 4 = \lambda^2 + \lambda - 2 = 0$$

$\lambda = -2$ and $\lambda = 1$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is unstable (saddle)

$\boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ $\begin{bmatrix} -3 & 2 \\ 2 & -2 \end{bmatrix} = DF\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$$P(\lambda) = (-3-\lambda)(-2-\lambda) - 4 = \lambda^2 + 5\lambda + 2 = 0$$

$$\lambda = \frac{-5 \pm \sqrt{25 - 4(2)}}{2} = \frac{-5 \pm \sqrt{17}}{2}$$

$\begin{bmatrix} 1 \end{bmatrix}$ is a stable sink

$$\boxed{\begin{bmatrix} 2 \\ -2 \end{bmatrix}} \quad \begin{bmatrix} -3 & -4 \\ 4 & 4 \end{bmatrix} = DF\left(\begin{bmatrix} 2 \\ -2 \end{bmatrix}\right)$$

$$P(\lambda) = (-3-\lambda)(4-\lambda)+16 = \lambda^2 - \lambda + 4 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1-16}}{2} = \frac{1 \pm i\sqrt{15}}{2}$$

$\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ is an unstable spiral

$$\boxed{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \quad \begin{bmatrix} 3 & -4 \\ 4 & 4 \end{bmatrix} = DF\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right)$$

$$P(\lambda) = (3-\lambda)(4-\lambda)+16 = \lambda^2 - 7\lambda + 28 = 0$$

$$\lambda = \frac{7 \pm \sqrt{49-4(28)}}{2}$$

$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is an unstable spiral

$\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ is an unstable spiral.

Example:

$$x' = \frac{1}{2}x - y - \frac{1}{2}(x^3 + y^2 x)$$

$$y' = x + \frac{1}{2}y - \frac{1}{2}(y^3 + x^2 y)$$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a fixed point

$$DF = \begin{bmatrix} \frac{1}{2} - \frac{3}{2}x^2 - \frac{y^2}{2} & -1 - xy \\ 1 - xy & \frac{1}{2} - \frac{3}{2}y^2 - \frac{x^2}{2} \end{bmatrix}$$

$$DF\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{bmatrix}$$

$$P(\lambda) = \left(\frac{1}{2} - \lambda\right)^2 + 1 \qquad \lambda = \frac{1}{2} \pm i$$

$\begin{bmatrix}0\\0\end{bmatrix}$ unstable spiral.

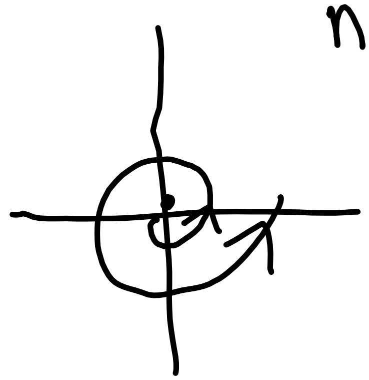
$\boxed{\lambda = \frac{1}{2} + i}$ $DF\left(\begin{bmatrix}0\\0\end{bmatrix}\right) - \left(\frac{1}{2} + i\right)I =$

$= \begin{bmatrix}-i & -1\\1 & -i\end{bmatrix}$ $v = \begin{bmatrix}i\\1\end{bmatrix}$

$v e^{\left(\frac{1}{2}+i\right)t} = e^{\frac{t}{2}} \begin{bmatrix}i\\1\end{bmatrix}(\cos t + i \sin t) =$

$= e^{t/2} \begin{bmatrix}-\sin t\\ \cos t\end{bmatrix} + i\, e^{t/2} \begin{bmatrix}\cos t\\ \sin t\end{bmatrix}$

near $\begin{bmatrix}0\\0\end{bmatrix}$



Change of variables

polar coordinates

$x = r\cos\theta \qquad r = r(t)$

$y = r\sin\theta \qquad \theta = \theta(t)$

$x' = r'\cos\theta - r\sin\theta\,\theta'$

$y' = r'\sin\theta + r\cos\theta\,\theta'$

(1) $r'\cos\theta - r\sin\theta\,\theta' = \frac{1}{2}r\cos\theta - r\sin\theta - \frac{r^3}{2}\cos\theta$

(2) $r'\sin\theta + r\cos\theta\,\theta' = r\cos\theta + \frac{r}{2}\sin\theta - \frac{r^3}{2}\sin\theta$

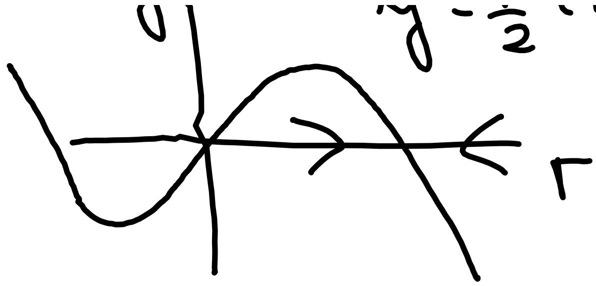
Multiply 1st eq by $\cos\theta$, multiply 2nd eqn by $\sin\theta$, add to get

$$\boxed{r' = \frac{1}{2}r - \frac{1}{2}r^3}$$

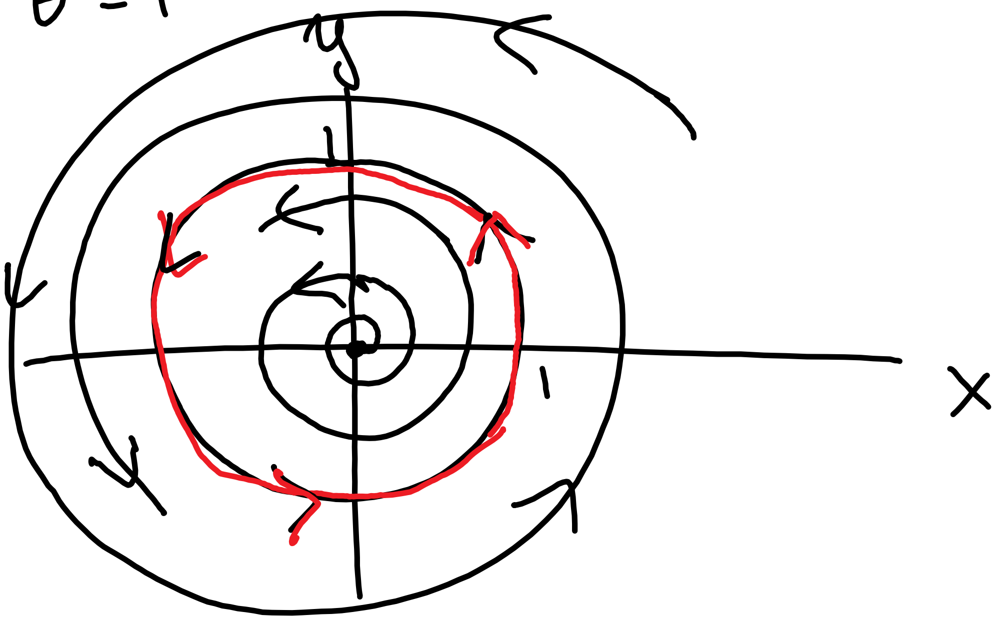
multiply (1) by $\left(\frac{-\sin\theta}{r}\right)$, mutiply (2) by $\frac{\cos\theta}{r}$, add to get

$$\boxed{\theta' = 1}$$

$$r' = \frac{r}{2}(1-r^2)$$



$$\theta' = 1$$



<u>Example</u>

$$x' = -y + \varepsilon x(x^2+y^2)$$

$$y' = x + \varepsilon y(x^2+y^2)$$

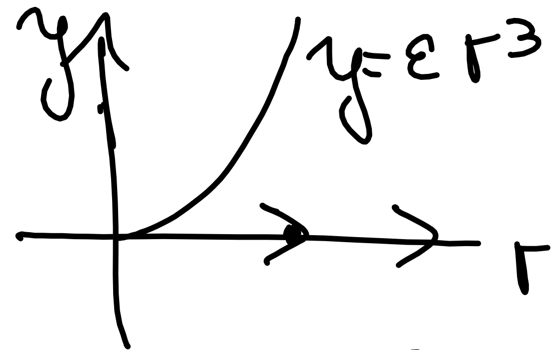
$\begin{bmatrix}0\\0\end{bmatrix}$ is a fixed point

$$DF\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$$

$$P(\lambda) = (-\lambda)^2+1 \qquad \lambda = \pm i$$

We can not conclude if $\begin{bmatrix}0\\0\end{bmatrix}$ is stable or unstable

In polar coordinates:

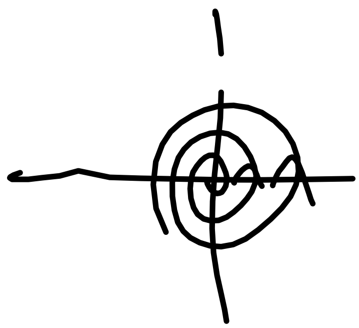


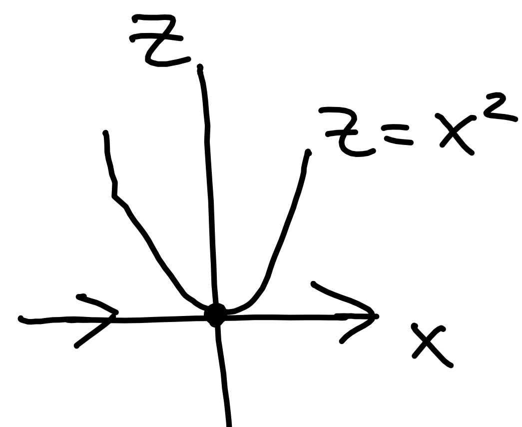
$r' = \varepsilon r^3$

$\theta' = 1$

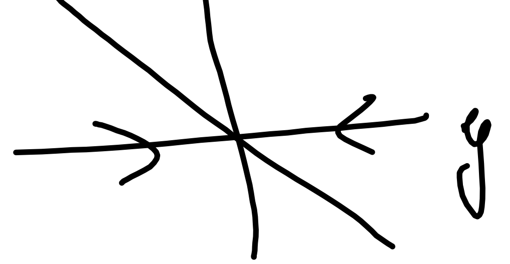
$r=0$ is unstable

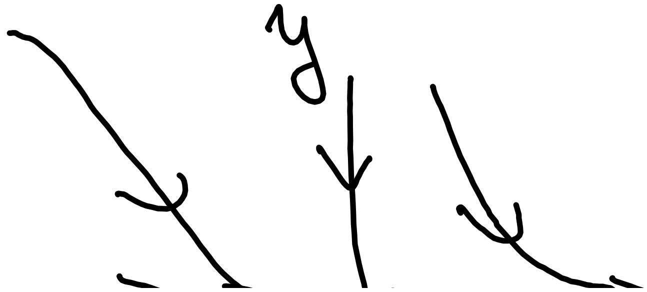
$\begin{bmatrix}0\\0\end{bmatrix}$ is an unstable spiral.

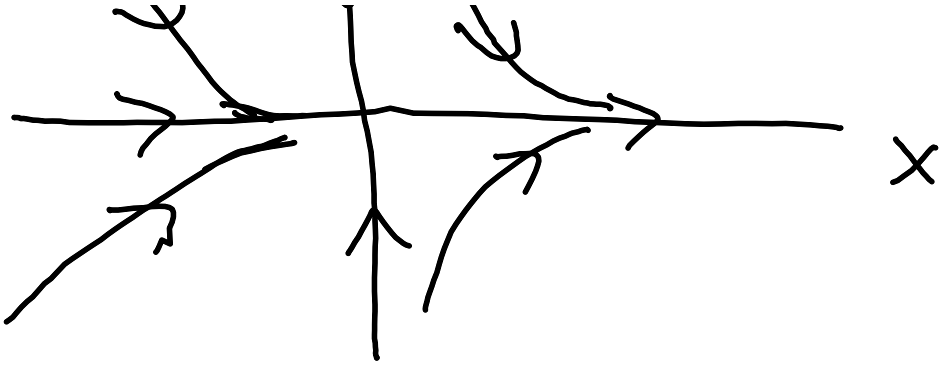




Example: $x' = x^2$
$y' = -y$



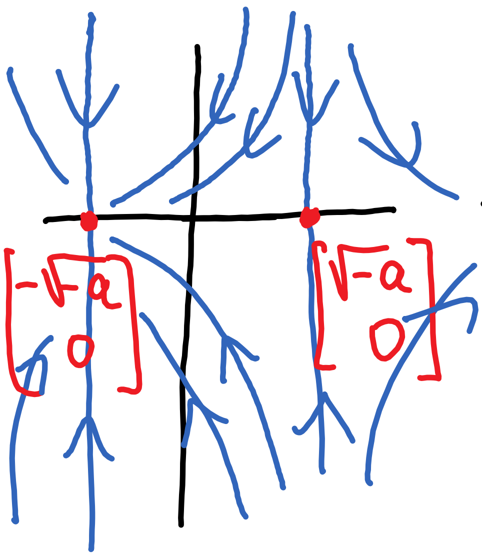


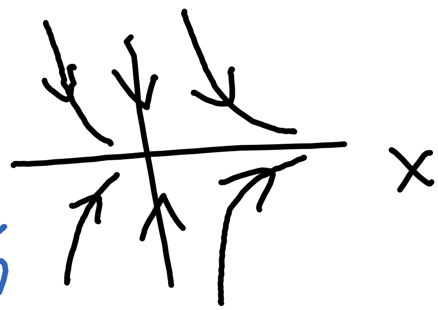


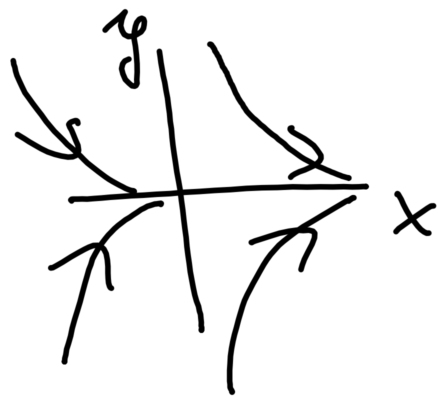
Example: $x' = x^2 + a$

$y' = -y$

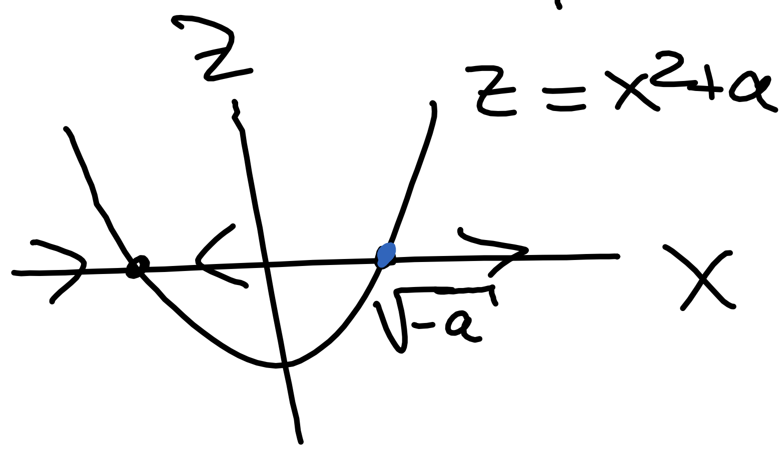
$a < 0$ $\qquad$ $a = 0$ $\qquad$ $a > 0$







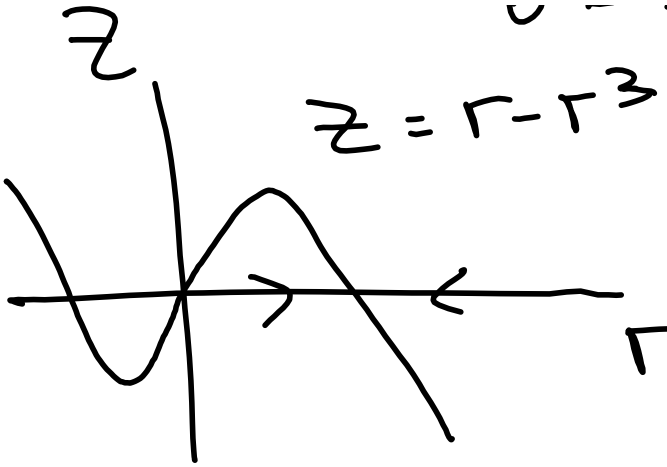
$x' = x^2 + a$

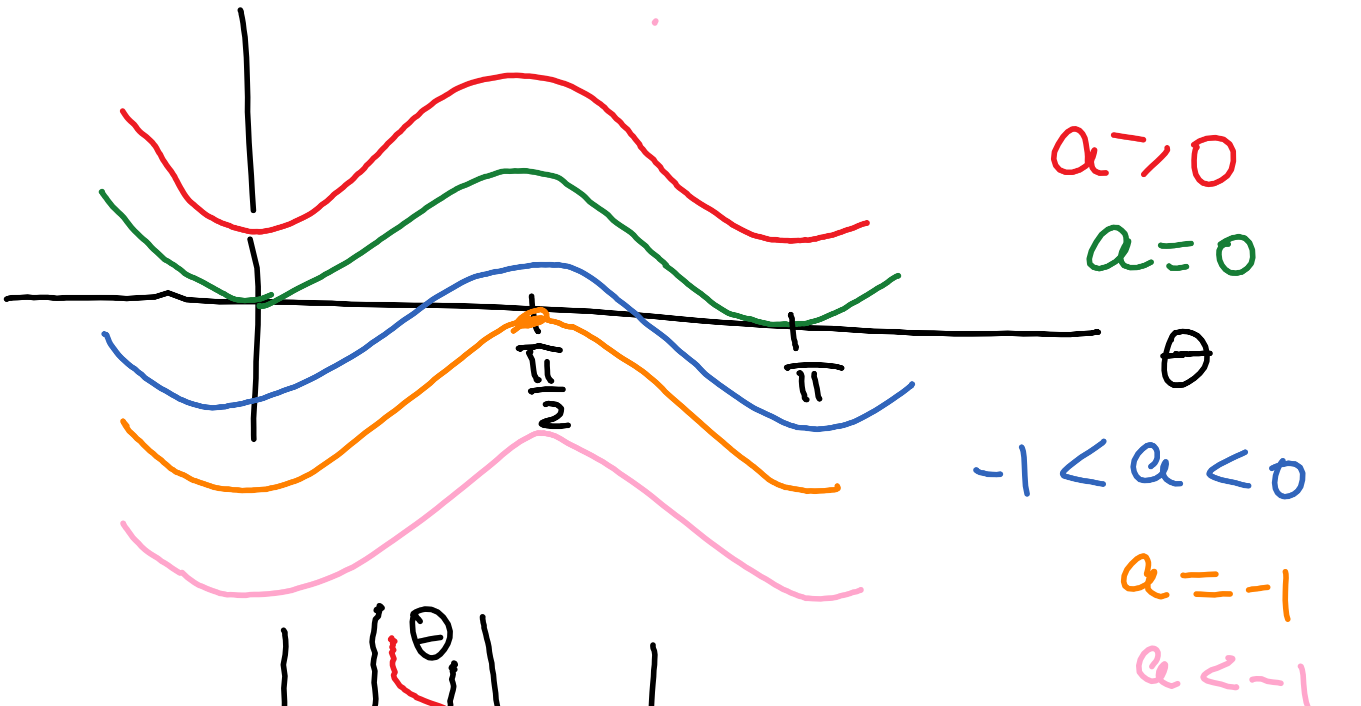


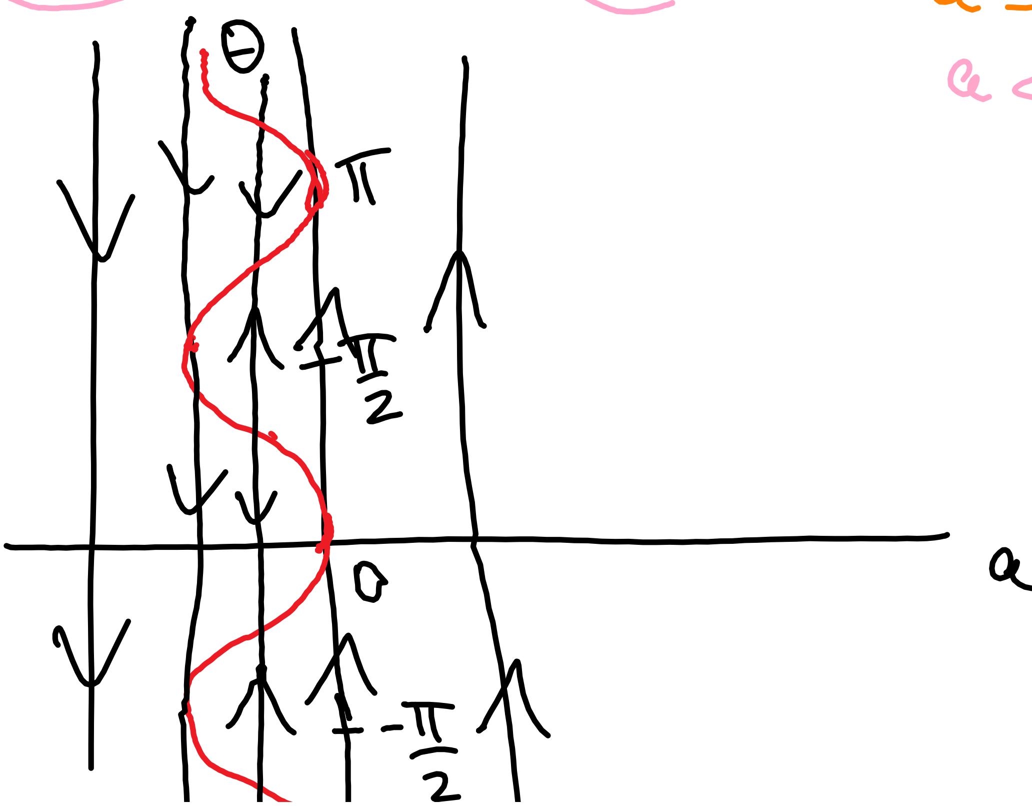
Example: $r' = r - r^3$

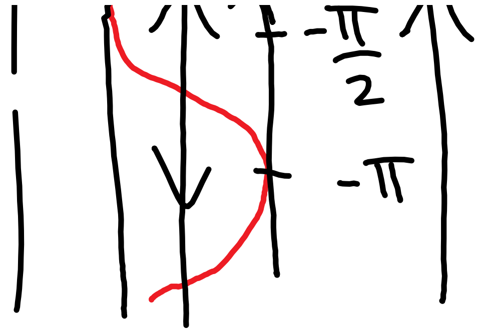
$\theta' = \sin^2\theta + a$

z' $-$ $-r^3$


z
$z = r - r^3$
r


$\frac{\pi}{2}$
π
θ
$a > 0$
$a = 0$
$-1 < a < 0$
$a = -1$
$a < -1$

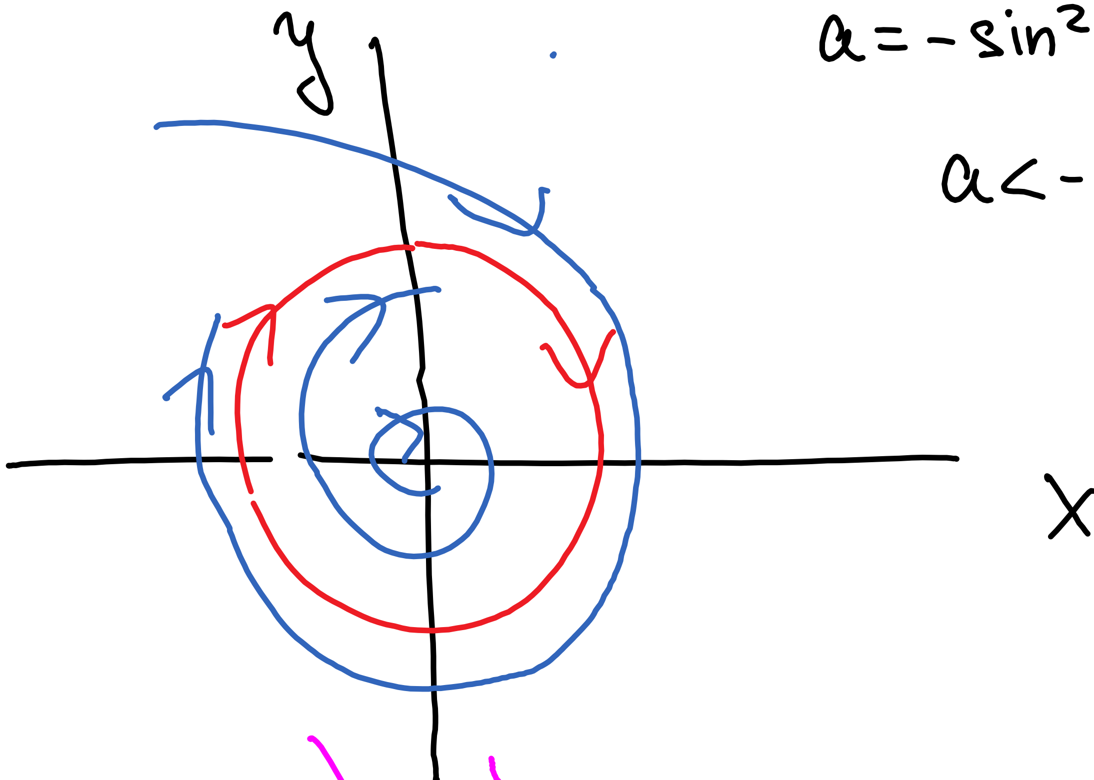

θ
π
$\frac{\pi}{2}$
0
$-\frac{\pi}{2}$
a



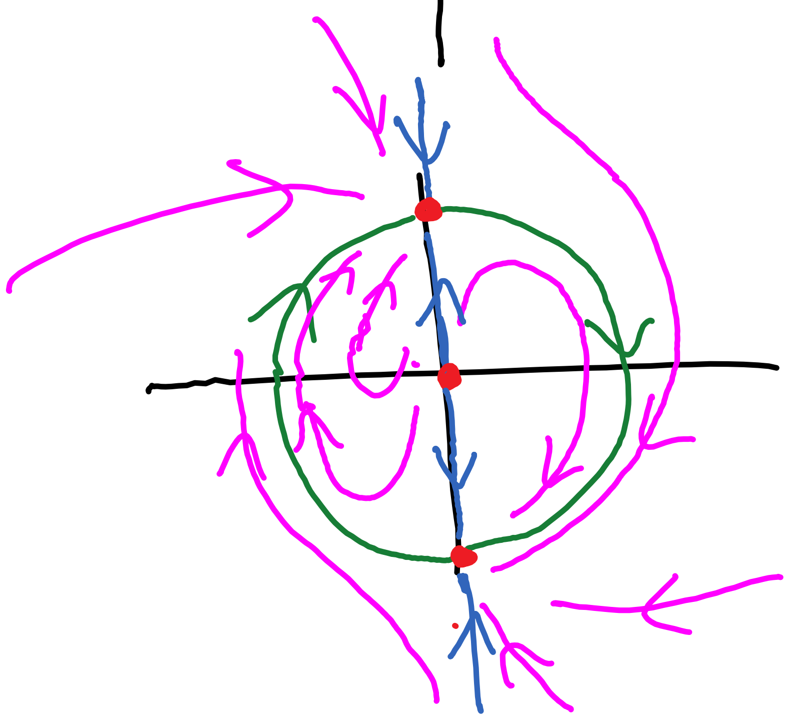
$\sin^2\theta + a = 0$

$a = -\sin^2\theta$

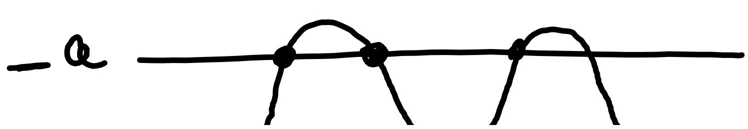
$a < -1$



$a = -1$

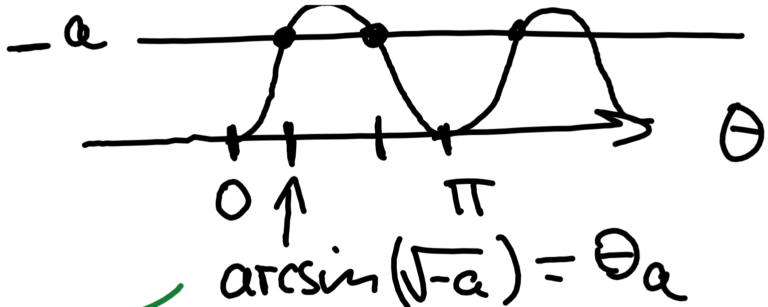


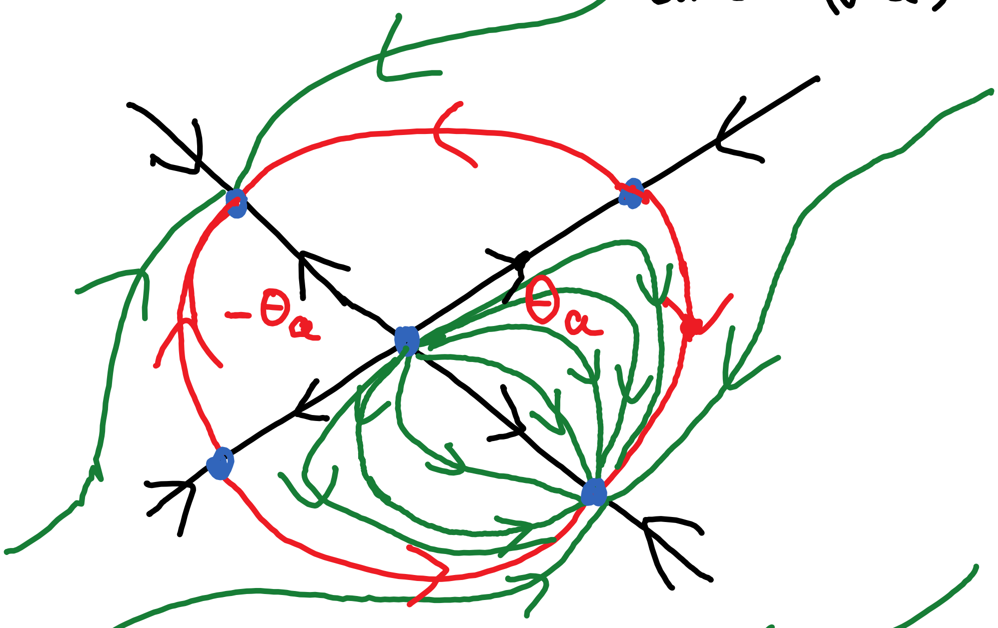
$-1 < a < 0$



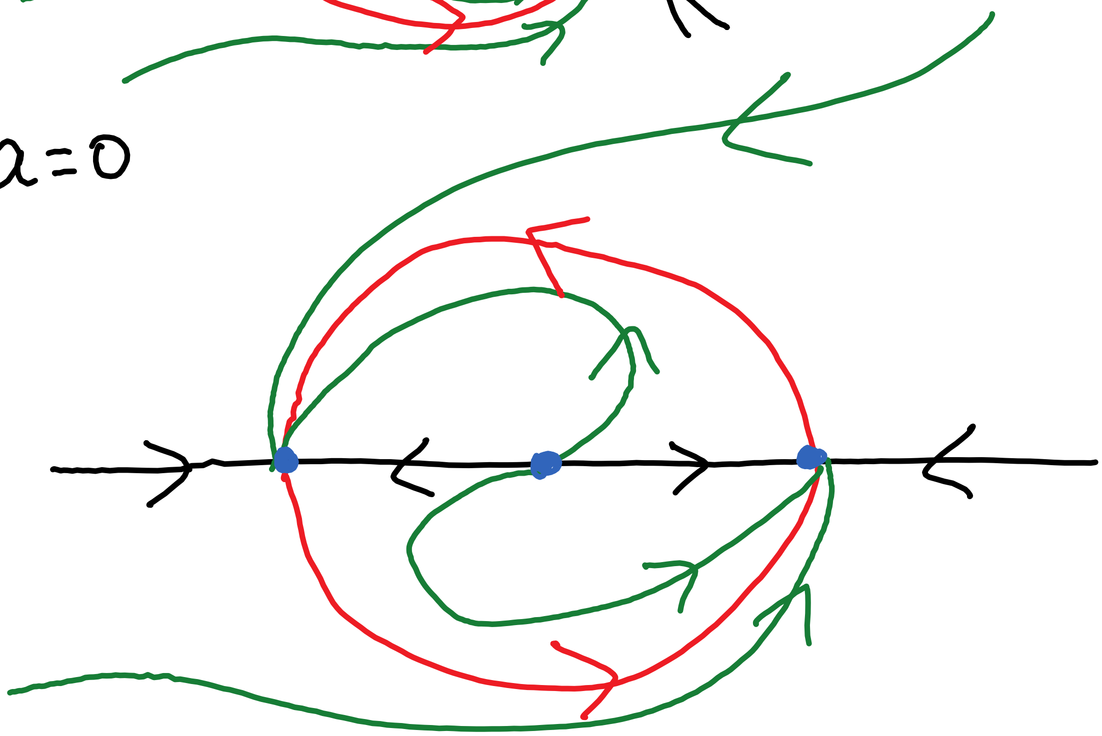
$-\frac{1}{2} < a < 0$

$\sin^2\theta + a = 0$





$a = 0$



<u>Ex:</u>

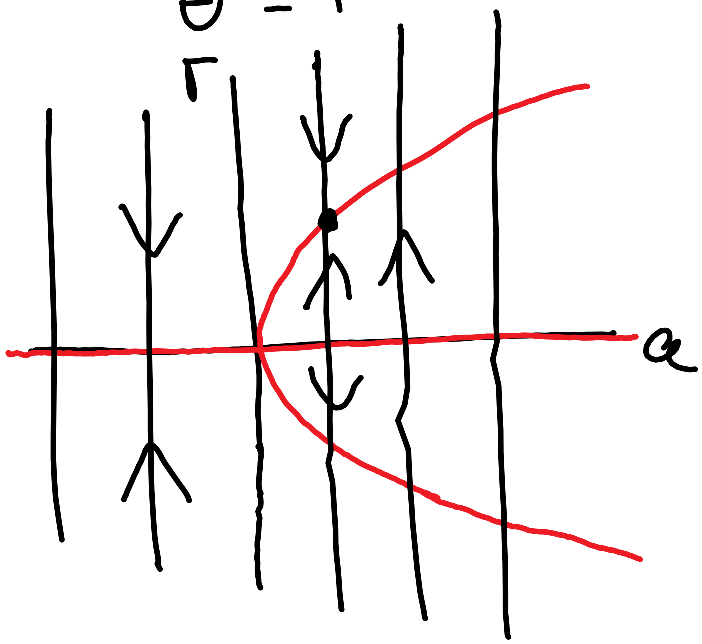
$$x' = ax - y - x(x^2+y^2)$$

$$y' = x + ay - y(x^2+y^2)$$

$x = r\cos\theta \qquad y = r\sin\theta$

$r' = ar - r^3$

$\theta' = 1$



$ar - r^3 = 0$

$a = r^2 \qquad r = 0$

$a \leq 0$

$a > 0$