4541_15

Tuesday, October 16, 2018 9:57 AM

Bifurcations

$x' = f(x, a)$ $f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,a) = f_a(x)$

What happens to fixed points as a changes?

x^* a fixed point.

$x^* = x^*(a)$

$0 = f(x^*, a) = f(x^*(a), a)$

replace a by $a + \Delta a$ and x^* by $x^* + \Delta x^*$

$0 = f(x^* + \Delta x^*, a + \Delta a)$

Linear approximation

Linear approximation:

$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + \frac{\partial f}{\partial x}(x,y)\Delta x + \frac{\partial f}{\partial y}(x,y)\,\Delta y$$

$$0 = f(x^* + \Delta x^*, a + \Delta a)$$

$$0 = \underbrace{f(x^*, a)}_{=0} + \frac{\partial f}{\partial x}(x^*, a)\,\Delta x + \frac{\partial f}{\partial a}(x^*, a)\,\Delta a$$

$$\boxed{\Delta x^* = -\frac{\frac{\partial f}{\partial a}(x^*, a)}{\frac{\partial f}{\partial x}(x^*, a)}\,\Delta a}$$

this is OK when $\frac{\partial f}{\partial x}(x^*, a) \neq 0$

$$\frac{\partial f}{\partial x}(x^* + \Delta x^*, a + \Delta a) \approx \frac{\partial f}{\partial x}(x^*, a)$$

<u>Theorem</u>: $x' = f(x,a) = f_a(x)$

Assume x^* is a fixed point when $a = a^*$.

(this means $f(x^*, a^*) = 0$).

Assume $f'_{a^*}(x^*) = \frac{\partial f}{\partial x}(x^*, a^*) \neq 0$.

Then, there exist $\varepsilon > 0$ and a unique continuous function $\Delta x^* : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ such that $\Delta x^*(0) = 0$ and

$$f(x^* + \Delta x^*(\Delta a), a + \Delta a) = 0$$

and, the sign of

$\frac{\partial f}{\partial x}(x^* + \Delta x^*(\Delta a), a + \Delta a) =$ to the sign of $\frac{\partial f}{\partial x}(x^*, a)$.

this means the fixed point

$x^* + \Delta x^*(\Delta a)$ when $a = a^* + \Delta a$

is of the same type as x^* when

is of the same type

$a = a^*$

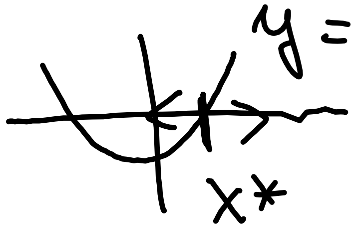
<u>Ex</u>: $x' = x^2 + a$

$a^* = -1 \qquad x^* = 1$

$f(x,a) = x^2 + a$

$\frac{\partial f}{\partial x}(1,-1) = 2 \neq 0 \qquad x^* = 1$ is a source

$y = x^2 - 1 \quad a^* = -1$



x^*

$$(x^* + \Delta x^*)^2 + (a^* + \Delta a) = 0$$

$$(1 + \Delta x^*)^2 + (-1 + \Delta a) = 0$$

$$(1 + \Delta x^*)^2 = 1 - \Delta a \qquad \varepsilon = 1$$

$$\Delta x^* = -1 \pm \sqrt{1 + \Delta a}$$

$$\Delta x^* = 0 \text{ if } \Delta a = 0$$

$\Delta x^* = -1 + \sqrt{1+\Delta a}$

then $\boxed{\Delta x^* = -1 + \sqrt{1 + \Delta a}}$

$1 + (-1 + \sqrt{1+\Delta a})$ is the fixed point when $a = -1 + \Delta a$, that corresponds (continuously) to the fixed point $x = 1$ when $a = -1$.

$1 + (-1 + \sqrt{1+\Delta a})$ is a source for all Δa such that $|\Delta a| < 1$.

there is no bifurcation in this case

Question: $x' = f_a(x) = f(x, a)$.

$f(x^*, a^*) = 0$. Also $\frac{\partial f}{\partial x}(x^*, a^*) = 0$.

Also $\frac{\partial f}{\partial a}(x^*, a^*) \neq 0$. Also $\frac{\partial^2 f}{\partial x^2}(x^*, a^*) \neq 0$

$x = x^* + \Delta x$

$$x = x^* + \Delta x$$

$$a = a^* + \Delta a$$

$$f(x,a) \approx \overset{=0}{\boxed{f(x^*, a^*)}} + \overset{=0}{\boxed{\frac{\partial f}{\partial x}(x^*, a^*)}} \Delta x + \frac{\partial f}{\partial a}(x^*, a^*) \Delta a + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x^*, a^*)(\Delta x)^2$$

$$f(x,a) \approx \frac{\partial f}{\partial a}(x^*, a^*) \Delta a + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x^*, a^*)(\Delta x)^2$$

$x = x^* + \Delta x$ then $x' = (\Delta x)'$

$$(\Delta x)' \approx \underbrace{\frac{\partial f}{\partial a}(x^*, a^*)}_{\alpha} \Delta a + \frac{1}{2} \underbrace{\frac{\partial^2 f}{\partial x^2}}_{\beta}(x^*, a^*)(\Delta x)^2$$

$$(\Delta x)' = \alpha \, \Delta a + \beta (\Delta x)^2$$

Do we still have a fixed point?

If yes, $\Delta x =$ constant and $\Delta x' = 0$

$$0 = \alpha(\Delta x)^2 + \beta \Delta a$$

$$\Delta x = \pm \sqrt{-\frac{\beta}{\alpha} \Delta a}$$

No fixed point if $-\frac{\beta}{\alpha} \Delta a < 0$

Two fixed points if $-\frac{\beta}{\alpha} \Delta a > 0$

In this case

$$x^* + \Delta x = x^* \pm \sqrt{\frac{-\frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x^*, a^*)}{\frac{\partial f}{\partial a}(x^*, a^*)} \Delta a}$$

In this case, $|\Delta x| = \text{conts} \sqrt{|\Delta a|}$

Then $|\Delta x|^2 = O(|\Delta a|)$

$$|\Delta x \Delta a| = O(|\Delta a|^{3/2}) << |\Delta a|$$

$$|\Delta a|^2 << |\Delta a|$$

Theorem: $x' = f_a(x)$. $f(x^*, a^*) = 0$.

$\frac{\partial f}{\partial x}(x^*, a^*) = 0$. $\frac{\partial^2 f}{\partial x^2}(x^*, a^*) \neq 0$.

$\frac{\partial f}{\partial a}(x^*, a^*) \neq 0$.

Let $c = -\frac{1}{2} \dfrac{\frac{\partial^2 f}{\partial x^2}(x^*, a^*)}{\frac{\partial f}{\partial a}(x^*, a^*)}$

If $c > 0$, then: $(\Delta a = a - a^*)$

If $\Delta a < 0$, there is no fixed point

If $\Delta a = 0$, there is one fixed point (x^*)

If $\Delta a > 0$, there are two fixed points

$$x = x^* + \Delta x = a^* \pm \sqrt{c}\,\sqrt{\Delta a}$$

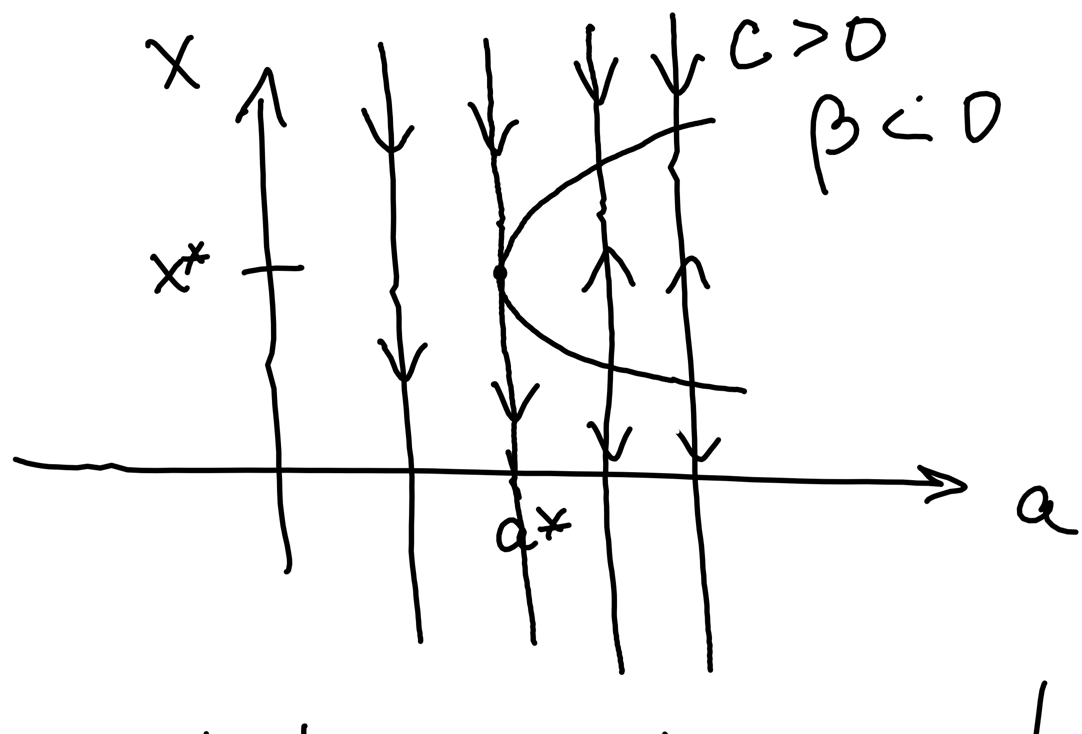
If $c < 0 \Rightarrow$ the opposite

2 fixed points $x = x^* + \Delta x = a^* \pm \sqrt{-c}$

$\sqrt{-\Delta a}$ if $\Delta a < 0$

no fixed point if $\Delta a > 0$

Bifurcation diagram



This is what is called a saddle node bifurcation.

this is the bifurcation we expect
in one-dimension